Universität Stuttgart

On the computation of Green functions

MIT Lie Groups Seminar

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Meinolf Geck

- Fix a prime p and a simple algebraic group G over $k = \overline{\mathbb{F}}_p$.
- Assume G defined over $\mathbb{F}_q \subseteq k$ $(q = p^f)$, with Frobenius map $F\colon G \to G$.
- Obtain finite group $G(\mathbb{F}_q) = G^F = \{g \in G \mid F(g) = g\}$.

$$|G(\mathbb{F}_q)| = q^{\dim G} + \text{smaller powers of } q.$$

General aim/program:

Determine $\mathrm{Irr}(G^F)$ = complex irreducible characters of G^F.

- Parametrisation, values on semisimple elements: solved (Lusztig 1980s).
- Arbitrary elements: theory of character sheaves (Lusztig 1984–today) ...
- ... where certain normalisations (roots of unity) remain to be determined.

And with this, create an electronic ATLAS of "generic" character tables, extending the famous Cambridge ATLAS of finite groups.

(E.g., the latter contains character table of $F_4(\mathbb{F}_2)$; would like this for $F_4(\mathbb{F}_{2^f})$, all $f \geqslant 1$.)

In this talk:

- Special case for above program: “Green functions”.
- “Normalisation problem” for Green functions solved in most cases, mainly by work of Beynon–Spaltenstein, Shoji and Lusztig (1980s–2000s).
- Will explain how to solve last remaining open cases, for G of exceptional types (uses computer calculations).

Platform for electronic ATLAS and computer calculations (ongoing project).

CHEVIE: `M.Geck, G.Hiss, F.Luebeck, G.Malle, J.Michel, G.Pfeiffer`

`http://www.math.rwth-aachen.de/~CHEVIE/`

`Implemented in` GAP3; `64-bit version, with many extensions, see:`

`http://webusers.imj-prg.fr/~jmichel/gap3`

`Latest development: Port to` Julia `language (J. Michel).`

Let $T \subseteq G$ be an F-stable maximal torus and $\theta \in \mathrm{Irr}(T^F)$

$$\rightsquigarrow \quad R_{T,\theta} \text{ virtual character of } G^F \quad \text{(Deligne and Lusztig 1970s).}$$

Let G_{uni} be the variety of unipotent elements of G

$$\rightsquigarrow \quad \text{Green function} \quad Q_T \colon G^F_{\mathrm{uni}} \to \overline{\mathbb{Q}}_\ell, \quad u \mapsto R_{T,\theta}(u).$$

- Q_T has values in $\mathbb{Z}$, and does not depend on θ.
- Character formula: Get all values of $R_{T,\theta}$ from Q_T and inductive procedure.

Lusztig 1984: Knowledge of all Q_T's $\rightsquigarrow$ "average value" character table of G^F.

Let $\rho \in \mathrm{Irr}(G^F)$ and C be an F-stable conjugacy class of G. Then C^F splits into finitely many classes in G^F, with representatives $g_1, \ldots, g_r \in C^F$ say.

$$\rightsquigarrow \quad \text{"average value"} \quad \mathrm{AV}(\rho, C) := \sum_{1 \leqslant i \leqslant r} [A_i : A_i^F]\, \rho(g_i),$$

where $A_i = C_G(g_i)/C_G^\circ(g_i)$ finite group (with induced action of F).

Assume from now on: G of adjoint type, defined and split over $\mathbb{F}_q$.

The G^F-conjugacy classes of F-stable maximal tori $T \subseteq G$ are parametrised by the conjugacy classes of W (Weyl group of G); write $T = T_w$ for $w \in W$.

- "New" Green functions $Q_\phi := |W|^{-1} \sum_{w \in W} \phi(w) Q_{T_w}$ for $\phi \in \mathrm{Irr}(W)$.
- Values of Q_ϕ are given by $m \times n$ table, where

  $m = |\mathrm{Irr}(W)|$ and $n =$ number of unipotent classes of G^F.

Example: G of type E_8.

$$m = |\mathrm{Irr}(W)| = 112 \qquad \text{and} \qquad n = \begin{cases} 146 & \text{if } p = 2, \\ 127 & \text{if } p = 3, \\ 117 & \text{if } p = 5, \\ 113 & \text{if } p > 5, \end{cases}$$ (Mizuno 1980)).

Above assumption implies: If C is a unipotent class of G, then $F(C) = C$ and there exists $u \in C^F$ such that F acts trivially on $A(u) = C_G(u)/C_G^\circ(u)$.

Values of Q_ϕ "*almost known*" by a general, purely combinatorial algorithm.

- For $\phi \in \mathrm{Irr}(W)$ there is a unique unipotent class $C = C_\phi$ of G such that
$$\{g \in G^F_{\mathrm{uni}} \mid Q_\phi(g) \neq 0\} \subseteq \overline{C} \qquad \text{and} \qquad Q_\phi|_{C^F} \neq 0.$$
Thus, obtain a map $\phi \mapsto C_\phi$ (= Springer correspondence).
- Set $d_\phi := (\dim G - \dim C_\phi - \mathrm{rank}(G))/2 \in \mathbb{Z}_{\geqslant 0}$. Define $Y_\phi \colon G^F_{\mathrm{uni}} \to \mathbb{Q}$ by
$$Q_\phi|_{C^F_\phi} = q^{d_\phi} Y_\phi \qquad \text{(and extend by 0 outside } C^F_\phi\text{)}.$$
- Lusztig, Shoji, ... (1976–2012): There are unique $p_{\phi',\phi} \in \mathbb{Z}$ such that
$$Q_\phi = \sum_{\phi' \in \mathrm{Irr}(W)} q^{d_\phi} p_{\phi',\phi} Y_{\phi'} \qquad \text{for all } \phi \in \mathrm{Irr}(W).$$
Matrix $(p_{\phi',\phi})$ is triangular with 1 on diagonal.
It can be computed by a purely combinatorial algorithm, which relies on a priori knowledge of the map $\phi \mapsto C_\phi$, i.e., Springer correspondence for G.
- $\rightsquigarrow$ Function `ICCTable` in J. Michel's version of CHEVIE.

"*Almost known*" ? Let $\phi \in \mathrm{Irr}(W)$, $C = C_\phi$, and Y_ϕ be the corresponding function. The remaining problem is to determine the values of Y_ϕ on C_ϕ^F.

- Let $u_\phi \in C_\phi^F$ be such that F acts trivially on $A(u_\phi) = C_G(u_\phi)/C_G^\circ(u_\phi)$.
- Springer correspondence also associates to ϕ a character $\psi_\phi \in \mathrm{Irr}(A(u_\phi))$.
- Let $a_1, \ldots, a_r \in A(u_\phi)$ be representatives of the conjugacy classes of $A(u_\phi)$.
- There are corresponding representatives $u_1, \ldots, u_r \in C_\phi^F$ of the conjugacy classes of G^F into which C_ϕ^F splits.

Then there exists a sign $\delta_\phi = \pm 1$ such that $\qquad Y_\phi(u_i) = \delta_\phi \psi_\phi(a_i) \qquad$ for all i.

$\rightsquigarrow$ *Everything is reduced to the — tricky! — task of determining the signs δ_ϕ.*

- For G of classical type, signs are determined by Shoji (1980s, 2007).
- For G of exceptional type, Beynon–Spaltenstein (1984), except for cases where p is small.
- For G_2, ${}^3D_4$, F_4, E_6, ${}^2E_6$ and p small, various explicit computations by Enomoto, Enomoto–Yamada, Spaltenstein, Malle, Porsch, Marcelo–Shinoda (1970s–1990s).

Open cases: ${}^2E_6(\mathbb{F}_{3^f})$, $E_7(\mathbb{F}_{2^f})$, $E_7(\mathbb{F}_{3^f})$, $E_8(\mathbb{F}_{2^f})$, $E_8(\mathbb{F}_{3^f})$, $E_8(\mathbb{F}_{5^f})$.

Theorem (G., 2020) Let $\phi \in \mathrm{Irr}(W)$ and δ_ϕ be the corresponding sign.

For $n \geqslant 1$, consider the group $G^{F^n} = G(\mathbb{F}_{q^n})$ and the Green function $Q_{\phi,n}\colon G^{F^n}_{\mathrm{uni}} \to \mathbb{Q}$, with corresponding sign $\delta_{\phi,n}$. Then we have $\delta_{\phi,n} = \delta_\phi^n$.

Proof uses interpretation of Green functions in terms of character sheaves, work of Lusztig and Shoji; and there are no restrictions on the characteristic p.

Theorem motivated by general character theory of finite groups. Let Γ, S be finite groups of coprime order such that S is solvable and acts by automorphisms on Γ.

Glauberman correspondence: $\mathrm{Irr}_S(\Gamma) \xleftrightarrow{1-1} \mathrm{Irr}(C_\Gamma(S))$, $\chi \leftrightarrow \chi^*$.

Problem/Conjecture (1990s): *The degree of χ^* divides the degree of χ.*

Hartley–Turull (1994): (1) Reduction to finite simple groups; (2) by classification: difficult case are groups of Lie type; and (3) for these, it is enough to show:

Congruence condition for Green functions.

Let $T \subseteq G$ be an F-stable maximal torus and $u \in G^F$ be unipotent. Let $r \in \mathbb{N}$ be a prime such that $r \nmid |G^{F^r}|$. Then $Q_{T,F}(u) \equiv Q_{T,F^r}(u) \bmod r$.

In G. 2020, this is deduced from the theorem; so, $\chi^*(1) \mid \chi(1)$ holds in general !

Back to problem of determining the signs δ_ϕ for $\phi \in \mathrm{Irr}(W)$.

Recall $G^F = G(\mathbb{F}_q)$ where $q = p^f$ with $f \geqslant 1$.

- Theorem implies that it is enough to compute δ_ϕ assuming $f = 1$.
- Hence, “only” need to compute values of Q_ϕ for the 6 individual groups

$$ {}^2E_6(\mathbb{F}_3), \quad E_7(\mathbb{F}_2), \quad E_7(\mathbb{F}_3), \quad E_8(\mathbb{F}_2), \quad E_8(\mathbb{F}_3), \quad E_8(\mathbb{F}_5). $$

Open case challenge: $G(\mathbb{F}_q) = E_8(\mathbb{F}_2)$; 112 functions Q_ϕ.

If character table of $E_8(\mathbb{F}_2)$ was known (like for other groups as in the Cambridge ATLAS), then we could easily determine the 112 missing signs.

But: $|E_8(\mathbb{F}_2)| \approx 3 \cdot 10^{73}$ and size of character table is 1156×1156

(see `http://www.math.rwth-aachen.de/~Frank.Luebeck/chev/index.html`)

Brute force methods won't work. Solved by alternative methods, as follows.

Recall: Q_ϕ's linear combinations of Y_ϕ's, Y_ϕ known up to δ_ϕ. Further information: Let $B \subseteq G$ be an F-stable Borel subgroup and $T_1 \subseteq B$ an F-stable maximal torus.

$$R_{T_1,1} = \text{permutation character of } G^F \text{ on cosets of } B^F.$$

So, if $u \in G^F$ is unipotent, then

$$\sum_{\phi \in \mathrm{Irr}(W)} \phi(1) Q_\phi(u) = Q_{T_1}(u) = R_{T_1,1}(u) = |\{gB^F \in G^F/B^F \mid g^{-1}ug \in B^F\}|.$$

Hence, if we can compute the right hand side, then we may get information on δ_ϕ's.

Example: $G^F = E_8(\mathbb{F}_q)$ with $q = p^f$ where $p \neq 3$

Let C = unipotent class $E_8(b_6)$ with representative $u = z_{77}$ (Mizuno 1980, $D_8(a_3)$).

We have $A(u) = C_G(u)/C_G^\circ(u) \cong \mathfrak{S}_3$ and $|C_G(u)^F| = 6q^{28}$.

There are three $\phi \in \mathrm{Irr}(W)$ such that $C_\phi = C$, denoted 2240_{10}, 175_{12} and 840_{13}.
(Springer correspondence known by Spaltenstein 1982, 1985.)

Want to determine $\delta_{2240_{10}} = \pm 1$, $\delta_{175_{12}} = \pm 1$, $\delta_{840_{13}} = \pm 1$.

Beynon–Spaltenstein (1984): If $p > 5$, then $\delta_{2240_{10}} = \delta_{175_{12}} = 1$, $\boldsymbol{\delta_{840_{13}} \equiv q}$ **mod 3**.

For $p \in \{2, 5\}$, run the GAP algorithm `ICCTable`. This yields the following identity:

$$|\{gB(\mathbb{F}_q) \in G(\mathbb{F}_q)/B(\mathbb{F}_q) \mid g^{-1}z_{77}g \in B(\mathbb{F}_q)\}| = R_{T_1,1}(z_{77}) = Q_{T_1}(z_{77})$$

$$= (2240q^{10}+3688q^9+3444q^8+2360q^7+1351q^6+672q^5+294q^4+112q^3 +35q^2+8q+1)\delta_{2240_{10}}+350q^{10}\delta_{175_{12}}+(840q^{10}+650q^9+160q^8)\delta_{840_{13}}$$

By Theorem, we only need to consider the cases where $q = p$.

For $q = p = 2$, obtain identity $\quad |\{gB(\mathbb{F}_2) \in G(\mathbb{F}_2)/B(\mathbb{F}_2) \mid g^{-1} z_{77} g \in B(\mathbb{F}_2)\}|$
$= 5{,}479{,}485\, \delta_{2240_{10}} + 358{,}400\, \delta_{175_{12}} + 1{,}233{,}920\, \delta_{840_{13}}$.

For $q = p = 5$, obtain identity $\quad |\{gB(\mathbb{F}_5) \in G(\mathbb{F}_5)/B(\mathbb{F}_5) \mid g^{-1} z_{77} g \in B(\mathbb{F}_5)\}|$
$= 30{,}631{,}220{,}541\, \delta_{2240_{10}} + 3{,}417{,}968{,}750\, \delta_{175_{12}} + 9{,}535{,}156{,}250\, \delta_{840_{13}}$.

In both cases, we can already conclude that $\delta_{2240_{10}} = 1$.

Now, total number of cosets $G(\mathbb{F}_p)/B(\mathbb{F}_p)$ is still huge, roughly $\sqrt{|G(\mathbb{F}_p)|}$.
So practically impossible to create actual permutation representation.

Consider Bruhat decomposition $\quad G(\mathbb{F}_p) = \coprod_{w \in W} B(\mathbb{F}_p) w B(\mathbb{F}_p)$;

each double coset $B(\mathbb{F}_p)wB(\mathbb{F}_p)$ contains exactly $p^{\ell(w)}$ cosets of $B(\mathbb{F}_p)$.

$\rightsquigarrow$ Systematic way of enumerating (in principle) all coset representatives for $G(\mathbb{F}_p)/B(\mathbb{F}_p)$, proceding by increasing $\ell(w)$.

Can use matrix realization of $G(\mathbb{F}_p)$ to perform these computations.
This is not very efficient but it was good enough to obtain:

Proposition (G. 2020). Re-compute, or compute for the first time, the signs δ_ϕ for all exceptional $G \neq E_8$ and $p = 2, 3$. In all these cases, we always have $\delta_\phi = 1$.

- Suggestions of F. Lübeck: Instead of matrices, work with Chevalley generators $x_\alpha(t)$ of G (where α is a root, $t \in k$) and commutator relations.
- Number of cosets fixed by an element can be obtained as number of $\mathbb{F}_p$-solutions of system of polynomial equations in several variables.
- $\rightsquigarrow$ Julia package `ChevLie1.1` (G.; independent GAP programs by Lübeck).

Back to E_8: For $p = 2$, number of cosets fixed by z_{77} should equal

$$5{,}479{,}485 + 358{,}400\,\delta_{175_{12}} + 1{,}233{,}920\,\delta_{840_{13}}.$$

With `ChevLie1.1`, find exact number of cosets fixed by z_{77} : $4{,}603{,}965$.
(This takes < 2 mins on my laptop.) Hence, $\delta_{175_{12}} = 1$ and $\delta_{840_{13}} = -1$.

For $p = 5$, number of cosets fixed by z_{77} should equal

$$30{,}631{,}220{,}541 + 3{,}417{,}968{,}750\,\delta_{175_{12}} + 9{,}535{,}156{,}250\,\delta_{840_{13}}.$$

With `ChevLie1.1`, find all 24,514,033,041 (!!!) cosets fixed by z_{77} in < 4 mins. (Note $[G(\mathbb{F}_5) : B(\mathbb{F}_5)] \approx 4 \cdot 10^{84}$.) Hence, again, $\delta_{175_{12}} = 1$ and $\delta_{840_{13}} = -1$.

Lübeck (2021) has now worked through all remaining classes in E_8 and $p = 2, 3, 5$.

Corollary. The Green functions are explicitly known in all cases (all G). The class $C = E_8(b_6)$ considered above is the only example where we can have $\delta_\phi = -1$.

Next steps:

- Compute Lusztig's generalised Green functions.
- Determine complete tables of values (not just average values) of unipotent characters of $G(\mathbb{F}_q)$, for G of exceptional type and all $q = p^f$, $f \geqslant 1$.