Graded Lie algebras, character sheaves & representations of double affine Hecke algebras

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Joint with K. Vilonen & partly with M. Grinberg

G reductive algebraic group $/\mathbb{C}$

$\theta: G \to G$ semisimple automorphism, order m (finite)

θ induces $\mathbb{Z}/m\mathbb{Z}$-grading on

$$\operatorname{Lie} G = \mathfrak{g} = \bigoplus_{i \in \mathbb{Z}/m} \mathfrak{g}_i .$$

(graded Lie algebras arise naturally from p-adic gps via Moy-Prasad filtration)

(classified by Kac, Vinberg, Reeder-Yu-Levy-Gross)

Let $K = (G^\theta)^\circ$ $(\mathrm{Lie}\, K = \mathfrak{g}_0)$ $\mathcal{N}$ = nilp cone of $\mathfrak{g}$

Vinberg: invariant theory of $K \curvearrowright \mathfrak{g}_i$ parallel to $G \curvearrowright \mathfrak{g}$

($m=2$: Kostant – Rallis: symmetric pairs (real gps)

In particular $K \curvearrowright \mathcal{N}_i := \mathfrak{g}_i \cap \mathcal{N}$ has finitely many orbits.

Def character sheaves on $\mathfrak{g}_1$

$\mathrm{Char}(\mathfrak{g}, \theta) := \{$ simple K-equivariant perverse sheaves on $\mathfrak{g}_1$ with nilpotent singular support$\}$

$= \{ \mathrm{Four}(IC(\mathcal{O}, \mathcal{E})) \mid \mathcal{O} \subset \mathcal{N}_{-1}$ K-orbit, $\mathcal{E}$ irr K-equiv local system on $\mathcal{O} \}$ (finite set)

by definition

"anti-orbital complexes"

Four : $\mathrm{Perv}_K(\mathfrak{g}_{-1})_{\mathbb{C}^*\text{-conic}} \xrightarrow{\sim} \mathrm{Perv}_K(\mathfrak{g}_1)_{\mathbb{C}^*\text{-conic}}$

$\cup$

$\mathrm{Perv}_K(\mathcal{N}_{-1})$

$\mathfrak{g}_1 \cong \mathfrak{g}_{-1}^*$ (via G-inv θ-inv nondeg bilinear form on $\mathfrak{g}$)

<u>Goal</u> Describe the set $\mathrm{char}(\mathfrak{g}, \theta)$

$= \{ IC(\text{support, local system}) \}$

This can be viewed as a Springer theory for graded Lie algebras, hope to be useful for repns of p-adic gps

<u>Rmk</u> 1) ungraded case $(m=1)$ Lusztig's generalised Springer correspondence ($G\times\cdots\times G$: θ permutes factors)

2) θ = inner $G = GL(n)$ Lusztig

$m=2$ $(G,K) = (SL_{2n}, Sp_{2n})$ Grinberg, Henderson, Lusztig

In 1) & 2), character sheaves are associated to irr. repns of Coxeter gps

3) θ involution $(m=2)$

Ginzburg, Grojnowski, general study of char. sheaves.

4) $m=2$ $(G,K)=(SL_N, SO_N)$ Chen-Vilonen-X.

G classical : Vilonen-X.

irr. reps of (finite) Hecke algebras of Coxeter gps (with parameters ± 1)

enter description of character sheaves.

Rely on geometric nearby cycle construction

(Grinberg-Vilonen-X.) (The method goes back to

Grinberg's thesis: microlocal geometry / stratified Morse theory)

+ counting argument (contribution from D. Stanton)

We get a complete answer in this case.

<u>strategy for general θ</u> (combinatorics / geometry more complicated)

1) classify the <u>cuspidal</u> charater sheaves, i.e. those that do not appear as (shifts of) direct summand of parabolic induction of charater sheaves on θ-stable Levi subgroups contained in proper θ-stable parabolic subgroups.

2) study parabolic induction of cuspidals on Levi's

From now on, we concentrate on <u>cuspidal charater sheaves</u> ⑦

Ungraded case / $\mathbb{Z}$-graded case (Lusztig)

very few cuspidal charater sheaves

"Fourier self-dual", nilpotent support, clean "bi-orbital"

Moreover, cuspidals on $\mathfrak{g}_1$ in $\mathbb{Z}$-graded case

come from cuspidals on $\mathfrak{g}$

$\mathbb{Z}/m\mathbb{Z}$-graded case: $m \gg 0$ similar to $\mathbb{Z}$-graded case

($\mathfrak{g}_1$ is nilpotent)

In general, we expect the following:

a) nilpotent support cuspidal character sheaves are rare, they come from classical cuspidals on $\mathfrak{g}$.

b) many cuspidal char sheaves with non-nilpotent support. often full support, i.e. support $= \mathfrak{g}_1$

Moreover, they all arise from a geometric nearby cycle construction (+ variations)

Cuspidal character sheaves

Fourier transform commutes with parabolic induction

{ cuspidal sheaves on $\mathcal{N}_{-1}$ } $\xleftrightarrow[\text{transform}]{\text{Fourier}}$ { cuspidal character sheaves on $\mathfrak{g}_1$ } $\xleftrightarrow[\text{cycle}]{\text{Nearby}}$ { irreps of Hecke algebras of complex reflection groups } (at roots of unity)

bijection

b)

a) Lusztig–Yun

c) KZ functor

(*)

{ finite dim irreps of trigonometric DAHAs (of Coxeter gps) }

{ full support irreps of rational DAHAs of Complex reflection groups }

e) Etingof

d) Koszul duality

{ finite dim irreps of rational DAHAs (of Coxeter gps) }

a) Lusztig – Yun $\{\text{simple } K\text{-equivariant perverse sheaves on } \mathcal{N}_{-1}\}$

$\leftrightarrow$ {irr. repns of trigonometric DAHAs.}

(Lusztig : $\mathbb{Z}$ – graded case)

$D_K(\mathcal{N}_{-1})$ has a block decomposition.

The blocks are (roughly) indexed by $(M, C, \mathcal{F})$

where M is a pseudo-Levi subgroup of (G, θ)

and $(C, \mathcal{F})$ is a cuspidal pair for M (in the sense of (ungraded) Lusztig's generalised Springer correspondence)

We call them Lusztig – Yun block.

e.g. principal block contains all $IC(\mathcal{O}, \mathbb{C})$

Fix a LY block ξ, Lusztig-Yun associates

a graded DAHA $\mathbb{H}^{\xi}_{c}$ with parameters $c = \{c_i\}$

[Same data as in Lusztig's "classification of unip
repns of simple p-adic gps I, II"]

Thm (conjectured by Lusztig-Yun, proved by W. Liu)

$\{$simple perverse sheaves in $D_K(\mathcal{N}_{-1})_{\xi}\}$

$\overset{\sim}{\longleftrightarrow}$ $\{$simple (integrable) $\mathbb{H}^{\xi}_{c,\frac{1}{m}}$ -modules$\}$
(with prescribed generalised eigenvalues
of the polynomial part given by the grading).

Conj / theorem (Z. Yun, C-C Tsai)

Under LY construction

$$\{\text{cuspidal sheaves in } D_K(\mathcal{N}_{-1})_\xi\} \xleftrightarrow{\sim} \{\text{finite diml irreps of } \mathbb{H}^{\xi}_{c,\frac{1}{m}}\}$$

(varying the gradings)

b) Nearby cycle construction

Vinberg: $\exists$ Cartan subspace $\mathfrak{a} \subset \mathfrak{g}_1$

s.t $\mathbb{C}[\mathfrak{g}_1]^K \cong \mathbb{C}[\mathfrak{a}]^{W_{\mathfrak{a}}}$ ← polynomial algebra

$$W_{\mathfrak{a}} = \frac{N_K(\mathfrak{a})}{Z_K(\mathfrak{a})} \quad \text{little Weyl group}$$

In general, $W_{\mathfrak{a}}$ is a complex reflection gp (classified by Shephard-Todd)

($m=2$, $W_{\mathfrak{a}}$ is a Coxeter group)

We say $\dim \mathfrak{a}$ is the rank of $\mathfrak{g}_1$.

Gradings that afford cuspidal char sheaves

1) ungraded ($m=1$) Type B_n : $2n+1$ = square

Lusztig: e.g. classical gps C_n : n = trigular number

D_n : $2n$ = square

2) $m=2$: G classical, (G, K) : split symmetric pair.

3) general θ

i) <u>stable gradings</u> : in the sense of invariant theory

$\exists \quad x \in \mathfrak{g}_1^{s.s}$ s.t $Z_K(x)$ is finite

These have been classified by Reeder-Levy-Yu-Gross; indexed by regular elliptic numbers of $W\sigma$ (σ = outer class of θ, they were motivated by repn theory of p-adic gps)

($m=2$ stable grading $\longleftrightarrow$ split symmetric pair)

ii) rank 0 gradings (e.g. $\mathbb{Z}$-gradings are rank 0)

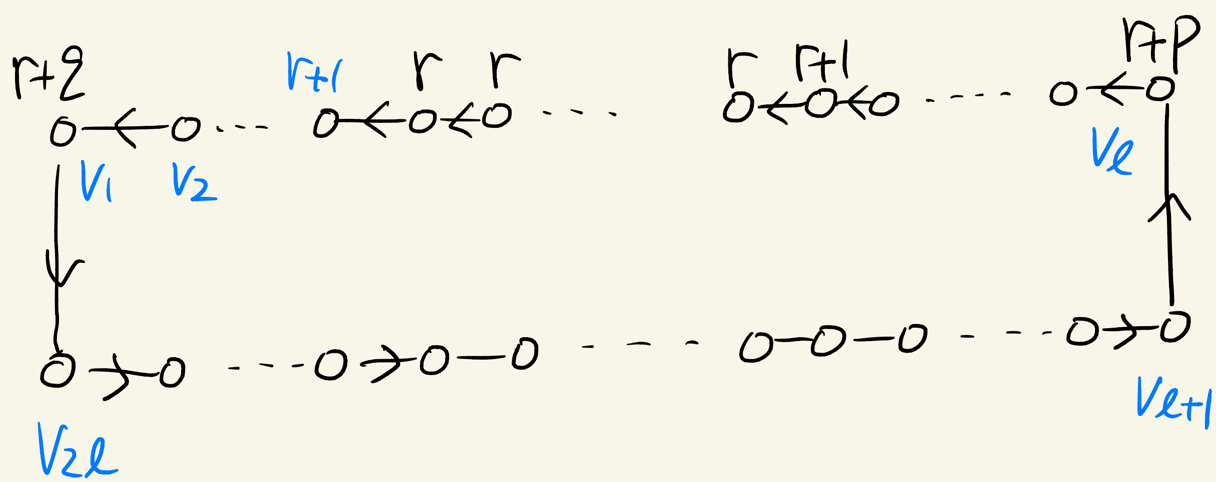
$(K, \mathfrak{g}_1)$
special prehomogeneous vector spaces (studied extensively, zeta fns)

<u>Conj</u> : G classical group. (suggested by diagram (*))

The gradings affording (non-nilp support) cuspidals are mixture of stable grading + special rank 0.

This can be made precise. We give an example here.

<u>Example</u> $G = Sp(2n)$ $m = 2\ell = \mathrm{order}(\theta)$



$$V = \bigoplus_{i=1}^{2\ell} V_i$$

$\langle V_i, V_j \rangle = 0$ unless $i + j = 2\ell + 1$

$G = Sp(V)$

$$K \cong \prod_{i=1}^{\ell} GL(V_i)$$

$$\mathfrak{g}_1 \cong \bigoplus_{i=2}^{\ell} \mathrm{Hom}(V_i, V_{i-1}) \oplus \mathrm{sym}^2(V_1) \oplus \mathrm{sym}^2(V_\ell^*)$$

$$W_a = G(m,1,r) = S_r \ltimes (\mathbb{Z}/m\mathbb{Z})^r$$

$r=0$: prehomogeneous vector space

Orbit has Jordan blocks $\quad$ local system

cuspidal $\quad 2+4+6+\cdots+2p \quad + \quad \mathcal{E}$ coming from

$2+4+6+\cdots+2q \quad$ Lusztig's cuspidal pair

$p=q=0$: stable grading

<u>Conj</u> (Work in progress) For $G = sp(2n)$ with grading as above, all cuspidal character sheaves arise from nearby cycle construction. They are of full support and correspond to irreps of Hecke algebras of $G(m,1,k) \times G(m,1,r-k)$ with Hecke relations of the form:

$$(T_s - 1)^2 = 0$$

$$(T_t - 1)^{\ell+p+2+1} (T_t + 1)^{\ell-p-2-1} = 0$$

$$(T_t - 1)^{\ell+p-2} (T_t + 1)^{\ell-p+2} = 0$$

B_n

$$\underbrace{\underset{1}{\circ} - \underset{1}{\circ} - \underset{1}{\circ}}_{S_n} = \underset{-1}{\circ}$$

<u>Rmk</u>: Hecke algebra associated to complex reflection group W introduced by Broué – Malle – Rouquier

It is free of rank $|W|$ (see Etingof's proof 2017)

Nearby cycles

Consider $f: \mathfrak{g}_1 \to \mathfrak{g}_1/\!\!/K \cong \mathfrak{a}/W_{\mathfrak{a}}$

Let $\bar{a} \in \mathfrak{a}^{r.s.}/W_{\mathfrak{a}}$ $\qquad$ $\mathfrak{a}^{r.s}$: regular semisimple locus.

We write $F_{\bar{a}} = f^{-1}(\bar{a})$

In general $F_{\bar{a}}$ is a finite union of K-orbits

we write $F_{\bar{a}}^{\circ} \subset F_{\bar{a}}$ the open dense K-orbit in $F_{\bar{a}}$

$F_{\bar{a}}^{ss} \subset F_{\bar{a}}$ the unique semisimple orbit in $F_{\bar{a}}$ (closed)

$$\mathfrak{g}_1^{reg} = \bigcup_{\bar{a} \in \mathfrak{a}^{rs}/W_{\mathfrak{a}}} F_{\bar{a}}^{\circ}$$

We have

$$\begin{tikzcd} F_{\bar{a}}^{\circ} \arrow[r, hook] \arrow[dr, "\varphi^{\circ}"'] & F_{\bar{a}} \arrow[d, "\varphi"] \\ & F_{\bar{a}}^{ss} \end{tikzcd}$$

$\forall\, b \in F_{\bar{a}}^{ss}$ $\quad Z_K(b) \circlearrowright \varphi^{\circ -1}(b)$ is a prehomogeneous vector space

Let $\mathcal{L}$ be a K-equivariant local system on $F_{\bar{a}}^{\circ}$ s.t its restriction to the fibers of φ° is rank 0 cuspidal local system

Take nearby cycle sheaf $\psi_f(IC(\mathcal{L})) := P_{\mathcal{L}}$

Grinberg's theorem $\Rightarrow$ $\mathrm{Four}(P_{\mathcal{L}}) = IC(M_{\mathcal{L}})$

- $\mathrm{rank}(M_{\mathcal{L}}) = |W_{\mathcal{L}}|$
- $P_{\mathcal{L}}$ has a large endomorphism group

To describe $M_{\mathcal{L}}$, we can reduce to the case of semisimple rank 1.

Let us now consider the rank 1 situation in the example, i.e. $r=1$

We have
$$\begin{array}{ccc} \mathfrak{g}_1 & \supset & \mathfrak{g}_1^{reg} = \{ f_1 \cdots f_{p+2}\, f \neq 0 \} \\ \downarrow f & & \\ \mathfrak{a}/W_{\mathfrak{a}} \cong \mathbb{C} & & \end{array}$$

Take $\mathcal{L}$ to be a rank 1 K-equiv. local system on $\mathfrak{g}_1^{reg}$ with -1 monodromy along all hypersurfaces $\{ f_i = 0 \mid i = 1, \cdots p+2 \}$

Then we claim

$$\mathrm{Four}(\Psi_f(IC(\mathcal{L}))) = IC\left(\mathfrak{g}_1^{reg*},\ \mathbb{C}_{\mathcal{L}} \otimes \frac{\mathbb{C}[x]}{(g(x))}\right)$$

Where
$$g(x) = \begin{cases} (x-1)^{l+p+2+1}(x+1)^{l-p-2-1} & \text{if } \mathcal{L} \text{ has trivial monodromy along } \{f=0\} \\ (x-1)^{l+p-2}(x+1)^{l-p+2} & \text{otherwise.} \end{cases}$$

Rmk 1) The above construction has been carried out in the case of stable polar repns (in our setting, this means $F^{\circ}_{\bar{a}} = F_{\bar{a}}$, i.e., the regular s.s orbits form a dense subset in $\mathfrak{g}_1$) in [Grinberg-Vilonen-X].

2) In [Vilonen-X.] we describe the local systems $M_{\mathcal{L}}$ explicitly in the case of stable gradings for classical types, in terms of Hecke algebras of the form discussed in the conjecture. (we make use of classification of perverse sheaves in the normal crossings case)

Consider the case of $r=1$, $p=q=0$ in the example
$(G = Sp(2l))$

we have $f: \mathfrak{g}_1 \cong \mathbb{C}^{\ell+1} \to \mathbb{C}$

$$f(x_1, \cdots, x_{\ell+1}) = x_1^2 \cdots x_{\ell-1}^2 \, x_\ell \, x_{\ell+1}$$

$$\pi_1^k(\mathfrak{g}_1^{rs}) = Z(G) \oplus \mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}$$

$$M_{\mathcal{L}} \cong \begin{cases} \dfrac{\mathbb{C}[x]}{(x^2-1)^{\ell-1}(x-1)^2} & \mathcal{L} \text{ trivial} \\ \mathbb{C}_{\mathcal{L}} \otimes \dfrac{\mathbb{C}[x]}{(x^2-1)^{\ell}} & \mathcal{L} \text{ nontrivial} \end{cases}$$

<u>Rmk</u> The conjecture can be formulated explicitly for all classical types.

c) KZ functor (Ginzburg – Guay – Opdam – Rouquier)

W – complex reflection group $\curvearrowright \mathfrak{h}$ – vector space

$H_c^{rat}(W)$ rational DAHA with parameter c $\quad \mathbb{C}[W] \ltimes (\mathbb{C}[\mathfrak{h}] \oplus \mathbb{C}[\mathfrak{h}^*])$

$\mathcal{O}(H_c^{rat}(W))$: category $\mathcal{O}$: $\mathfrak{h}$ acts locally nilp, f.g over $\mathbb{C}[\mathfrak{h}]$

$$\begin{tikzcd} \mathcal{O}(H_c^{rat}(W)) \arrow[dr] \arrow[dd, "KZ"'] & \\ & B_W\text{-mod} \arrow[dl] \\ H_{W,q}\text{-mod} & \end{tikzcd}$$

$B_W = \pi_1(\mathfrak{h}^{reg}/W)$

$H_{W,q} = \dfrac{\mathbb{C}[B_W]}{\text{Hecke relations}}$ $\qquad q = e^{2\pi i c}$

$KZ : \mathcal{O}/\mathcal{O}_{tor} \xrightarrow{\sim} H_{W,q}\text{-mod}$ $\qquad$ preserves blocks

$\mathcal{O}_{tor}$: modules supported on discriminant locus

d) Koszul duality of blocks of category $\mathcal{O}$ of (cyclotomic) rational DAHAs

(conj'd by Chuang – Miyachi, proved by RSVV)

$$\underset{\text{block}}{\mathcal{O}_t}\left(H^{\text{rat}}_{\underbrace{\frac{1}{e},s}_{\text{parameter}}}(G(l,1,r))\right) \overset{\text{Koszul}}{\longleftrightarrow} \mathcal{O}_s\left(H^{\text{rat}}_{\frac{1}{l},t}(G(e,1,r'))\right)$$

On the categorical level, there is derived equivalence

s.t. tilting $\longleftrightarrow$ simple module

$$\begin{array}{ccc} \cup & & \cup \\ \{\text{full support mod}\} & \longleftrightarrow & \{\text{fin. diml mod}\} \end{array}$$

expected to be a bijection (Shan, Losev)

Conj: arrow (d) in the diagram (*) is [RSVV] duality.

<u>Rmk</u>: In the case of exceptional gps, the diagram (*) suggests duality for exceptional type rational DAHAs.

e) Etingof:

trig. DAHA

$$\{\text{f.d irreps of } \mathbb{H}^{\xi}_{c,\frac{1}{m}}\} \xleftrightarrow{\sim} \coprod_{x} \{\text{f.d. irreps of } H^{rat}_{c,\frac{1}{m}}(W_x)\}$$

(analogy: Lusztig: affine Hecke alg $\longleftrightarrow$ graded Hecke algs)

<u>Conj</u> Restricting to the LY blocks, a) b) d) in diagram (*) are bijections.

(Fourier transform "=" Koszul duality)

<u>Rmks</u> 1) We expect all f.d irreps of rational DAHAs occur in this picture

2) repns of p-adic gps?

3) categorical explanation of diagram (*)?