Harmonic Analysis and Gamma Functions on Symplectic Groups

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Preliminaries

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- $\pi \simeq \bigotimes_p \pi_p \in \mathcal{A}_{\text{cusp}}(G)$;
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- $\pi \cong \bigotimes_p \pi_p \in A_{\text{cusp}}(G)$;
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- $\rho : {}^L G \to \text{GL}(V_\rho)$;
- $\pi \simeq \bigotimes_p \pi_p \in \mathcal{A}_{\text{cusp}}(G)$;
- According to R. Langlands, one should be able to define
  \[ L(s, \pi, \rho) = \prod_p L(s, \pi_p, \rho); \]
- By Langlands, $L(s, \pi, \rho)$ (actually the partial $L$-function) is absolutely convergent for $\text{Re}(s)$ large;
Langlands’ conjecture

$L(s, \pi, \rho)$ has a meromorphic continuation to $s \in \mathbb{C}$, and the functional equation

$$L(1 - s, \pi^\vee, \rho) = \varepsilon(s, \pi, \rho)L(s, \pi, \rho)$$

holds where $\varepsilon(s, \pi, \rho)$ is non-zero entire in $s \in \mathbb{C}$.

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- Methods: Godement-Jacquet (Tate), Rankin-Selberg; Langlands-Shahidi; Trace formula;
Preliminaries

Natural question
Establish the basic analytic properties for $L(s, \pi, \rho)$ through harmonic analysis on $G$ (or related spherical varieties).
R. Godement and H. Jacquet established the M.C. and F.E. of the standard $L$-function $L(s, \pi)$ of $\text{GL}_n$ (over $F$-central simple algebras) via harmonic analysis on $\text{GL}_n \hookrightarrow M_n$, generalizing the work of Tate for $n = 1$ (when $n = 2$ it was also done in the last chapter of Jacquet-Langlands).
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- $G = \text{GL}_n$;
- $L^G = \text{GL}_n(\mathbb{C}) \times \mathcal{W}_F$, $\rho = \text{Id} \otimes \{\text{trivial}\}$. 
For convenience, let \( p \) be a non-archimedean place of \( F \).

**Ingredients**

- Schwartz space \( S(G(F_p)) = C_c^\infty(M_n(F_p))\big|_{G(F_p)} \);
For convenience, let $p$ be a non-archimedean place of $F$.

**Ingredients**

- Schwartz space $S(G(F_p)) = C^\infty_c(M_n(F_p))|_{G(F_p)}$;
- Fourier transform $\mathcal{F}_{\psi_p} : S(G(F_p)) \to S(G(F_p))$;
For \( f \in S(G(F_p)) \), set

\[
Z(s, f, \varphi_{\pi_p}) = \int_{G(F_p)} f(g)\varphi_{\pi_p}(g)|\det g|_{F_p}^{s+\frac{n-1}{2}} dg, \quad s \in \mathbb{C},
\]

where \( \varphi_{\pi_p} \in C(\pi_p) \) (the space of matrix coefficients of \( \pi_p \)).
Godement-Jacquet: Local theory

Theorem (Godement-Jacquet)

$Z(s, f, \varphi_{\pi_p})$ is absolutely convergent for $\text{Re}(s)$ sufficiently large, and is a rational function in $q^{-s}$;
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Theorem (Godement-Jacquet)

- \( Z(s,f,\varphi_{\pi_p}) \) is absolutely convergent for \( \text{Re}(s) \) sufficiently large, and is a rational function in \( q^{-s} \);
- the set \( \{ Z(s,f,\varphi_{\pi_p}) \mid f \in S(G(F_p)), \varphi_{\pi_p} \in C(\pi_p) \} \) is a fractional ideal of \( \mathbb{C}[q^{-s}, q^s] \) with generator \( \frac{1}{P(q^{-s})} \), where \( P(q^{-s}) \) is a polynomial with \( P(0) = 1 \). Set \( L(s, \pi_p) = \frac{1}{P(q^{-s})} \).
Theorem (Godement-Jacquet)

- $Z(s, f, \varphi_{\pi_p})$ is absolutely convergent for $\Re(s)$ sufficiently large, and is a rational function in $q^{-s}$;

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- there exists a rational function $\gamma(s, \pi_p, \psi_p)$ in $q^{-s}$ such that the following functional equation holds for any $f \in S(G(F_p))$

\[
Z(1 - s, \mathcal{F}_{\psi_p}(f), \varphi_{\pi_p}^\vee) = \gamma(s, \pi_p, \psi_p)Z(s, f, \varphi_{\pi_p}).
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Theorem (Godement-Jacquet)

- \( \mathcal{Z}(s, f, \varphi_{\pi_p}) \) is absolutely convergent for \( \text{Re}(s) \) sufficiently large, and is a rational function in \( q^{-s} \);
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\mathcal{Z}(1 - s, \mathcal{F}_{\psi_p}(f), \varphi_{\pi_p}^\vee) = \gamma(s, \pi_p, \psi_p) \mathcal{Z}(s, f, \varphi_{\pi_p}).
\]

- Let \( 1_p \) be the characteristic function of \( M_n(\mathfrak{o}_p) \subset M_n(F_p) \). Then \( \mathcal{F}_{\psi_p}(1_p) = 1_p \) and \( \mathcal{Z}(s, 1_p, \varphi_{\pi_p}) = L(s, \pi_p) \) for any unramified representation \( \pi_p \) and \( \varphi_{\pi_p} \) zonal spherical.
Ingredients

- Schwartz space $\mathcal{S}(G(\mathbb{A})) = \bigotimes'_p \mathcal{S}(G(F_p))$ w.r.t. $\{1_p\}_{p<\infty}$;
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- Schwartz space $S(G(\mathbb{A})) = \bigotimes_p' S(G(F_p))$ w.r.t. $\{1_p\}_{p<\infty}$;
- Fourier transform $F_\psi = \bigotimes_p F_{\psi_p}$;
- For $f \in S(G(\mathbb{A}))$, consider

$$Z(s, f, \varphi_\pi) = \int_{G(\mathbb{A})} f(g)\varphi_\pi(g)|\det g|_{\mathbb{A}}^{s+\frac{n-1}{2}} d^\times g, \quad s \in \mathbb{C},$$

where $\varphi_\pi \in \mathcal{C}(\pi)$. 
Theorem (Godement-Jacquet)

When $\Re(s)$ is sufficiently large, $Z(s, f, \varphi_\pi)$ is absolutely convergent, and $Z(s, f, \varphi_\pi) = \prod_p Z_p(s, f_p, \varphi_{\pi_p})$ whenever $f = \otimes_p f_p$ is a pure tensor.
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- $\mathcal{Z}(s, f, \varphi_\pi)$ has a meromorphic continuation to $s \in \mathbb{C}$, and the functional equation

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holds.

- Meromorphic continuation and functional equation follow from the Poisson summation formula for $(S(G(\mathbb{A})), \mathcal{F}_\psi)$. 
Braverman-Kazhdan proposal

Around 2000, A. Braverman and D. Kazhdan proposed a conjectural framework to establish the analytical properties of general automorphic $L$-functions $L(s, \pi, \rho)$.

The prototype of the proposal is the theory of Godement and Jacquet.

For convenience, make the following additional assumptions (can be removed)

Assumptions

- $G/F$ split;
- $\rho$ is obtained from an irreducible injective representation of $G^\vee(\mathbb{C})$ with highest weight $\lambda_\rho$;
- $\sigma : G \rightarrow \mathbb{G}_m$ a character playing the role of det for $GL_n$;
Braverman-Kazhdan proposal: Local

For convenience, let $p$ be a non-archimedean place of $F$.

Conjectural ingredients

- Schwartz space $\mathcal{C}_c^\infty(G(F_p)) \subset S_{\rho}(G(F_p)) \subset \mathcal{C}^\infty(G(F_p))$;
For convenience, let $p$ be a non-archimedean place of $F$.

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- Schwartz space $\mathcal{C}_c^\infty(G(F_p)) \subset S_\rho(G(F_p)) \subset \mathcal{C}^\infty(G(F_p))$;
- Fourier transform $\mathcal{F}_{\rho,\psi_p} : S_\rho(G(F_p)) \to S_\rho(G(F_p))$. 
Braverman-Kazhdan proposal: Local

Setup

- For $f \in S_{\rho}(G(F_p))$, set

$$Z(s, f, \varphi_{\pi_p}) = \int_{G(F_p)} f(g)\varphi_{\pi_p}(g)|\sigma(g)|^{s+n_\rho}_{F_p} dg, \quad s \in \mathbb{C},$$

where $\varphi_{\pi_p} \in \mathcal{C}(\pi_p)$

- For geometric reason, may set $n_{\rho} = \langle \rho_B, \lambda_{\rho} \rangle$ where $\rho_B$ is the half sum of positive roots (Bouthier-Ngo-Sakellaridis).

- In general different $n_{\rho}$ differ by unramified shift;
Braverman-Kazhdan proposal: Local

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Expectation

- $Z(s, f, \varphi_{\pi_p})$ is absolutely convergent for $\text{Re}(s)$ sufficiently large and is a rational function in $q^{-s}$;
Braverman-Kazhdan proposal: Local

**Expectation**

1. \( Z(s, f, \varphi_{\pi_p}) \) is absolutely convergent for \( \Re(s) \) sufficiently large and is a rational function in \( q^{-s} \);

2. The set \( \{ Z(s, f, \varphi_{\pi_p}) \mid f \in S_{\rho}(G(F_p)), \varphi_{\pi_p} \in C(\pi_p) \} \) is a finitely generated fractional ideal in \( \mathbb{C}(q^{-s}) \) with generator \( L(s, \pi_p, \rho) \);
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Expectation

- $Z(s, f, \varphi_{\pi_p})$ is absolutely convergent for $\text{Re}(s)$ sufficiently large and is a rational function in $q^{-s}$;
- The set $\{Z(s, f, \varphi_{\pi_p}) \mid f \in S_{\rho}(G(F_p)), \varphi_{\pi_p} \in C(\pi_p)\}$ is a finitely generated fractional ideal in $\mathbb{C}(q^{-s})$ with generator $L(s, \pi_p, \rho)$;
- There exists a rational function $\gamma(s, \pi_p, \rho, \psi_p)$ in $q^{-s}$ such that the following functional equation holds for any $f \in S_{\rho}(G(F_p))$
  
  $Z(1 - s, F_{\rho, \psi_p}(f), \varphi_{\pi_p}^\vee) = \gamma(s, \pi_p, \rho, \psi_p)Z(s, f, \varphi_{\pi_p})$

  where $\varphi_{\pi_p} \in C(\pi_p)$;
Braverman-Kazhdan proposal: Local

Schwartz space

For any \((G, \rho)\), there is an affine spherical embedding \(G \hookrightarrow M_\rho\), where \(M_\rho\) arises from the theory of reductive monoids studied by M. Putcha, L. Renner and E. Vinberg. It is expected that \(S_\rho(G(F_p))\) is connected with the geometry of \(M_\rho\).
Braverman-Kazhdan proposal: Local Schwartz space

- For any \((G, \rho)\), there is an affine spherical embedding \(G \hookrightarrow \mathcal{M}_\rho\), where \(\mathcal{M}_\rho\) arises from the theory of reductive monoids studied by M. Putcha, L. Renner and E. Vinberg. It is expected that \(S_\rho(G(F_p))\) is connected with the geometry of \(\mathcal{M}_\rho\);

- There should exist \(\mathbb{L}_{\rho,p} \in S_\rho(G(F_p))^{K_p \times K_p}\) called the basic function, such that \(\mathcal{Z}(s, \mathbb{L}_{\rho,p}, \varphi_{\pi_p}) = L(s, \pi_p, \rho)\) for any unramified representation \(\pi_p\) and \(\varphi_{\pi_p}\) zonal spherical;
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- There should exist $L_{\rho,p} \in S_\rho(G(F_p))^{K_p \times K_p}$ called the basic function, such that $Z(s, L_{\rho,p}, \varphi_{\pi_p}) = L(s, \pi_p, \rho)$ for any unramified representation $\pi_p$ and $\varphi_{\pi_p}$ zonal spherical;

- For Godement-Jacquet, $M_\rho = M_n, L_{\rho,p} = 1_p$. 
Braverman-Kazhdan proposal: Local

Fourier transform

For any \( f \in \mathcal{C}_c^\infty(G(F_p)) \),

\[
\mathcal{F}_{\rho,\psi_p}(f)(g) = |\sigma(g)|^{-2n-1} (\Phi_{\rho,\psi_p} \ast f^\vee)(g);
\]

where \( \Phi_{\rho,\psi_p} \) is an invariant distribution on \( G(F_p) \) such that

\[
\Phi_{\rho,\psi_p}(\pi) = \gamma(\cdot, \pi, \rho, \psi_p) \cdot \text{Id}_\pi;
\]
Braverman-Kazhdan proposal: Local Fourier transform

- For any $f \in C_c^\infty(G(F_p))$,
  
  $$\mathcal{F}_{\rho,\psi_p}(f)(g) = |\sigma(g)|^{-2n\rho-1}(\Phi_{\rho,\psi_p} * f^\vee)(g);$$

  where $\Phi_{\rho,\psi_p}$ is an invariant distribution on $G(F_p)$ such that
  
  $$\Phi_{\rho,\psi_p}(\pi) = \gamma(\cdot, \pi, \rho, \psi_p) \cdot \text{Id}_\pi;$$

- $\mathcal{F}_{\rho,\psi_p}$ extends to a unitary operator on $L^2(G(F_p), |\sigma(\cdot)|^{2n\rho+1} dg)$ and $\mathcal{F}_{\rho,\psi_p} \circ \mathcal{F}_{\rho,\psi_p}^{-1} = \text{Id};$
Braverman-Kazhdan proposal: Local

Fourier transform

- For any $f \in C_c^\infty(G(F_p))$, 

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where $\Phi_{\rho,\psi_p}$ is an invariant distribution on $G(F_p)$ such that 

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- For Godement-Jacquet, $\Phi_{\rho,\psi_p}(g) = \psi(\text{tr}(g))|\det(g)|^n$. 
Braverman-Kazhdan proposal: Local unramified Theorem (L.)

For $\mathfrak{p}$ non-archimedean,

$$S_\rho(G(F_\mathfrak{p}))^{K_\mathfrak{p} \times K_\mathfrak{p}} = \mathbb{L}_{\rho,\mathfrak{p}} \ast C^\infty_c(G(F_\mathfrak{p}))^{K_\mathfrak{p} \times K_\mathfrak{p}}$$

and

$$\Phi^{K_\mathfrak{p}}_{\rho,\psi_\mathfrak{p}} = \text{Inverse Satake transform of } \gamma(-s - n_\rho, \pi_\mathfrak{p}, \rho^\vee, \psi_\mathfrak{p}).$$

The proposal is verified in full detail in unramified setting;
Braverman-Kazhdan proposal: Local unramified Theorem (L.)

- For $p$ non-archimedean,

$$S_\rho(G(F_p))^{K_p \times K_p} = \mathbb{I}_{\rho,p} \ast C_\infty(G(F_p))^{K_p \times K_p}$$

and

$$\Phi_{\rho,\psi_p}^{K_p} = \text{Inverse Satake transform of } \gamma(-s - n_\rho, \pi_p, \rho^\vee, \psi_p).$$

The proposal is verified in full detail in unramified setting;

- For $p$ archimedean, take $\mathbb{I}_{\rho,p}$ as the inverse Harish-Chandra transform of $L(s, \pi_p, \rho)$, then

$$\mathbb{I}_{\rho,p,s} = \mathbb{I}_{\rho,p}|\sigma(\cdot)|^s, \text{ and } \Phi_{\rho,\psi_p}^{K_p} = \Phi_{\rho,\psi_p}^{K_p}|\sigma(\cdot)|^s$$

can be plugged into the Arthur-Selberg trace formula when $\text{Re}(s)$ large.
Braverman-Kazhdan proposal: Global

Conjectural ingredients

- Schwartz space $\mathcal{S}_\rho(G(\mathbb{A})) = \bigotimes'_p \mathcal{S}_\rho(G(F_p))$ w.r.t. $\mathbb{L}_{\rho,p}$, $p < \infty$;
Braverman-Kazhdan proposal: Global

Conjectural ingredients

- Schwartz space \( S^\rho(G(\mathbb{A})) = \bigotimes'_p S^\rho(G(F_p)) \) w.r.t. \( \{\mathbb{L}_{\rho,p}\}_{p<\infty} \);
- Fourier transform \( \mathcal{F}^\rho,\psi = \bigotimes_p \mathcal{F}^\rho,\psi_p \).
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Conjectural ingredients

- Schwartz space \( S_\rho(G(\mathbb{A})) = \bigotimes'_p S_\rho(G(F_p)) \) w.r.t. \( \{\mathbb{I}_{\rho,p}\}_{p<\infty} \);
- Fourier transform \( \mathcal{F}_{\rho,\psi} = \bigodot_p \mathcal{F}_{\rho,\psi_p} \);
- \( \rho\)-Poisson summation formula for \( (S_\rho(G(\mathbb{A})), \mathcal{F}_{\rho,\psi}) \).
The work of Jiang-Luo-Zhang

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The work of Jiang-Luo-Zhang

- It is the first substantial case after the work of Godement-Jacquet;
- Establish the analytical theory of $L(s, \pi, \rho)$ following the approach of Godement-Jacquet, provide new evidence substantially for the Braverman-Kazhdan proposal.
The work of Jiang-Luo-Zhang

In the following, let $F$ be a $p$-adic field.

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- It is closely related to the doubling method of Piatetski-Shapiro and Rallis, the work of Lapid-Rallis, and other more recent works;
- The major work we need is the right normalization of the local intertwining operators appearing in doubling method;
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Doubling method (Piatetski-Shapiro and Rallis)

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Doubling method (Piatetski-Shapiro and Rallis)

- $(F^{2n}, \langle \cdot, \cdot \rangle)$;
- $\text{Sp}_{2n}$;

$P = MN = \text{Stab}(L \Delta) a$ Siegel parabolic in $\text{Sp}_{4n}$, where $L \Delta = \{ (v, v) | v \in F^{2n} \}$ is a Lagrangian;

$\text{Sp}_{2n} \times \text{Sp}_{2n} \hookrightarrow \text{Sp}_{4n} \to P \setminus \text{Sp}_{4n}$ has Zariski open dense image, with stabilizer $P \cap (\text{Sp}_{2n} \times \text{Sp}_{2n}) = \text{Sp}_{\Delta 2n} \hookrightarrow \text{Sp}_{2n} \times \text{Sp}_{2n}$;
Doubling method (Piatetski-Shapiro and Rallis)

- 
  - $(F^{2n}, \langle \cdot, \cdot \rangle)$;
  - $\text{Sp}_{2n}$;
  - $\text{Sp}_{2n} \times \text{Sp}_{2n} \hookrightarrow \text{Sp}_{4n}$ via $(F^{2n} \oplus F^{2n}, \langle \cdot, \cdot \rangle \oplus -\langle \cdot, \cdot \rangle)$;

- 
  - $P = MN = \text{Stab}(L \Delta)$ a Siegel parabolic in $\text{Sp}_{4n}$, where $L \Delta = \{(v, v) | v \in F^{2n}\}$ is a Lagrangian;
  - $\text{Sp}_{2n} \times \text{Sp}_{2n} \hookrightarrow \text{Sp}_{4n}$ has Zariski open dense image, with stabilizer $P \cap (\text{Sp}_{2n} \times \text{Sp}_{2n}) = \text{Sp}_{\Delta 2n} \hookrightarrow \text{Sp}_{2n} \times \text{Sp}_{2n}$;
Doubling method (Piatetski-Shapiro and Rallis)

- $(F^{2n}, \langle \cdot, \cdot \rangle)$;
- $\text{Sp}_{2n}$;
- $\text{Sp}_{2n} \times \text{Sp}_{2n} \hookrightarrow \text{Sp}_{4n}$ via $(F^{2n} \oplus F^{2n}, \langle \cdot, \cdot \rangle \oplus -\langle \cdot, \cdot \rangle)$;
- $P = MN = \text{Stab}(L_\Delta)$ a Siegel parabolic in $\text{Sp}_{4n}$, where $L_\Delta = \{(v, v) | v \in F^{2n}\}$ is a Lagrangian;
Doubling method (Piatetski-Shapiro and Rallis)

$$(F^{2n}, \langle \cdot, \cdot \rangle);$$

$${\mathcal{S}p}_2n;$$

$${\mathcal{S}p}_{2n} \times {\mathcal{S}p}_{2n} \hookrightarrow {\mathcal{S}p}_{4n} \text{ via } (F^{2n} \oplus F^{2n}, \langle \cdot, \cdot \rangle \oplus -\langle \cdot, \cdot \rangle);$$

$$P = MN = \text{Stab}(L_{\Delta}) \text{ a Siegel parabolic in } {\mathcal{S}p}_{4n}, \text{ where }$$

$$L_{\Delta} = \{(v, v) | \ v \in F^{2n}\} \text{ is a Lagrangian};$$

$${\mathcal{S}p}_{2n} \times {\mathcal{S}p}_{2n} \hookrightarrow {\mathcal{S}p}_{4n} \rightarrow P \backslash {\mathcal{S}p}_{4n} \text{ has Zariski open dense image, with stabilizer }$$

$$P \cap ({\mathcal{S}p}_{2n} \times {\mathcal{S}p}_{2n}) = {\mathcal{S}p}_{2n}^{\Delta} \hookrightarrow {\mathcal{S}p}_{2n} \times {\mathcal{S}p}_{2n};$$
The work of Jiang-Luo-Zhang

The following diagram illustrates the basic idea behind our work

\[
\begin{array}{ccc}
\mathbb{S}p_{4n} & \xrightarrow{\mathcal{M}^{\text{ab}} wN} & X_P \\
\downarrow & & \downarrow \\
\mathcal{M}^{\text{ab}}(\mathbb{S}p_{2n} \times \{I_{2n}\}) & \cong & \mathbb{G}_m \times \mathbb{S}p_{2n}
\end{array}
\]

where \(X_P = [P, P] \backslash \mathbb{S}p_{4n}\), \(w = (\text{Id}_{2n}, -\text{Id}_{2n}) \in \mathbb{S}p_{2n} \times \mathbb{S}p_{2n}\), \(\mathcal{M}^{\text{ab}} = [M, M] \backslash M \cong \mathbb{G}_m\).

- \(wPw = P^-, \mathcal{M}^{\text{ab}} wN\) is Zariski open dense in \(X_P\);
- \(G = \mathbb{G}_m \times \mathbb{S}p_{2n}\) is Zariski open dense in \(X_P\);
Harmonic analysis on $M^{ab} \hookrightarrow X_P$

Fourier transform

For $f \in \mathcal{C}^\infty_c(X_P(F))$, define

$$\mathcal{F}_{X,\psi}(f)(g) := \int_{F \times N(F)} \eta_{pvs,\psi}(x) |x|^{-\frac{2n+1}{2}} \int_{N(F)} f(wns(x)g) dndx.$$ 

where $s : \mathbb{G}_m \rightarrow M$ is a section of $M \rightarrow [M, M] \backslash M \cong \mathbb{G}_m$;
Harmonic analysis on $M^\mathbf{ab} wN \hookrightarrow X_P$

Fourier transform

- For $f \in C^\infty_c(X_P(F))$, define

  $$\mathcal{F}_{X,\psi}(f)(g) := \int_{F^\times} \eta_{\text{pvs},\psi}(x)|x|^{-\frac{2n+1}{2}} \int_{N(F)} f(wns(x)g)dndx.$$

  where $s : \mathbb{G}_m \to M$ is a section of $M \to [M, M]\backslash M \simeq \mathbb{G}_m$;

- $\eta_{\text{pvs},\psi}(x)$ is a distribution on $F^\times$, which is a key ingredient towards the understanding of $\mathcal{F}_\rho,\psi$ and $S_\rho(G(F))$;
Harmonic analysis on $M_{ab \to X_P}$

Fourier transform

- For $f \in C_\infty^c(X_P(F))$, define

$$\mathcal{F}_{X,\psi}(f)(g) := \int_{F^\times} \eta_{pvs,\psi}(x)|x|^{-\frac{2n+1}{2}} \int_{N(F)} f(wns(x)g)dndx.$$  

where $s : \mathbb{G}_m \to M$ is a section of $M \to [M, M]\backslash M \simeq \mathbb{G}_m$;

- $\eta_{pvs,\psi}(x)$ is a distribution on $F^\times$, which is a key ingredient towards the understanding of $\mathcal{F}_{\rho,\psi}$ and $S_\rho(G(F))$;

- The definition of $\eta_{pvs,\psi}$ first appeared in [Braverman-Kazhdan, 2002], but that definition of $\eta_{pvs,\psi}$ did not carry enough analytical information for our work.
Abelian harmonic analysis

- To understand the analytical nature of $\eta_{p_{\text{vs}},\psi}$, we develop the local harmonic analysis associated to $\eta_{p_{\text{vs}},\psi}$ in the spirit of Braverman-Kazhdan proposal;
Abelian harmonic analysis

- To understand the analytical nature of $\eta_{pvs,\psi}$, we develop the local harmonic analysis associated to $\eta_{pvs,\psi}$ in the spirit of Braverman-Kazhdan proposal;

- An explicit formula for $\eta_{pvs,\psi}$ is obtained from the functional equation associated to zeta integrals on the prehomogeneous space $(GL_{2n+1}, S_{2n+1})$, where $S_{2n+1}$ is the space of $(2n + 1) \times (2n + 1)$ symmetric matrices. More precisely, for a character $\chi$, the following zeta integral is considered

$$Z(s, f, \chi) = \int_{S_{2n+1}(F)} f(X)\chi(X)|\det X|^{s-(n+1)}dX;$$
Abelian harmonic analysis

- To understand the analytical nature of $\eta_{\text{pvs}, \psi}$, we develop the local harmonic analysis associated to $\eta_{\text{pvs}, \psi}$ in the spirit of Braverman-Kazhdan proposal;

- An explicit formula for $\eta_{\text{pvs}, \psi}$ is obtained from the functional equation associated to zeta integrals on the prehomogeneous space $(\text{GL}_{2n+1}, S_{2n+1})$, where $S_{2n+1}$ is the space of $(2n + 1) \times (2n + 1)$ symmetric matrices. More precisely, for a character $\chi$, the following zeta integral is considered

$$Z(s, f, \chi) = \int_{S_{2n+1}(F)} f(X)\chi(X)|\det X|^{s-(n+1)}dX;$$

- The functional equation for the zeta integrals on $(\text{GL}_m, S_m)$ is known by the work of Piatetski-Shapiro and Rallis, and T. Ikeda.
Abelian harmonic analysis

The following diagram illustrates the idea

\[ C_c^\infty(S_{2n+1}) \xrightarrow{\text{Fourier transform}} C_c^\infty(S_{2n+1}) \]

\[ \xrightarrow{\text{F.I.}} \]

\[ S^+_{n,\beta}(F^\times) \xrightarrow{|.|^{-2n}} S^+_{\text{pvs}}(F^\times) \xrightarrow{\mathcal{L}=\mathcal{L}_{\eta_{\text{pvs}},\psi}} S^-_{\text{pvs}}(F^\times) \xleftarrow{|.|^{n+1}} S^-_{n,\beta}(F^\times) \]

where

- \( \text{F.I.} \) is the fiber integration along \( \det : S_{2n+1} \to F \);
- \( \mathcal{L} \) is the induced linear transform.
Abelian harmonic analysis

Theorem (JLZ)

- $\mathcal{L}$ is well-defined;
Abelian harmonic analysis

Theorem (JLZ)

- $\mathcal{L}$ is well-defined;
- $S_{pvs}^+(F^\times)$ consists of functions $f$ in $C^\infty(F^\times)$, such that
  1. $\text{supp}(f)$ is bounded, i.e. $f(x) = 0$ for $|x| \gg 0$;
Abelian harmonic analysis

Theorem (JLZ)

- $\mathcal{L}$ is well-defined;
- $S_{pvs}^+(F^\times)$ consists of functions $f$ in $C^\infty(F^\times)$, such that
  1. $\text{supp}(f)$ is bounded, i.e. $f(x) = 0$ for $|x| \gg 0$;
  2. for $|x| \ll 1$,

\[
f(x) = a_0^+(ac(x))|x|^{-2n} + \sum_{i=0}^{n-1} a_{i,+}^+(ac(x))|x|^{i-\frac{2n-1}{2}} + a_{i,-}^+(ac(x))|x|^{i-\frac{2n-1}{2}} (-1)^{\text{ord}(x)}
\]

where $a_0^+$ is a locally constant function on $\mathfrak{o}_F^\times$ that is $\mathfrak{o}_F^\times$-invariant, $a_i^+$ are locally constant functions on $\mathfrak{o}_F^\times$ that are $\mathfrak{o}_F^{\times 2}$-invariant, $ac(x) = \frac{x}{|x|}$;
Abelian harmonic analysis

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where $a_0^+$ is a locally constant function on $\mathfrak{o}_F^\times$ that is $\mathfrak{o}_F^\times$-invariant, $a_{i,\pm}^+$ are locally constant functions on $\mathfrak{o}_F^\times$ that are $\mathfrak{o}_F^{\times 2}$-invariant, $ac(x) = \frac{x}{|x|}$;

- In particular, $C^\infty_c(F^\times) \hookrightarrow S^+_{pvs}(F^\times)$ is of finite codimension;
Paley-Wiener theorem for $S_{pvs}^{\pm}(F^\times)$

Theorem (JLZ)

- Under Mellin transform $(\int_{F^\times} f(x)\chi_s(x)dx)$, $S_{pvs}^+(F^\times)$ is captured by

$$L(s, \chi) \prod_{i=0}^{n-1} L(2s + 2i + 1, \chi^2).$$

It follows from the description of G.C.D. for the zeta integral $\mathcal{Z}(s, \cdot, \chi)$ attached to $(\text{GL}_m, S_m)$, which is established in our work (for $\chi$ unramified it is proved by Piatetski-Shapiro and Rallis).
Proposition (JLZ)

For any $f \in S_{\mathsf{pvs}}^+(F^\times)$, there is the following functional equation after meromorphic continuation

$$
\int_{F^\times} \mathcal{L}(f) \chi_{s + \frac{n+1}{2}}(t)^{-1} dt = \beta_\psi(\chi_s) \int_{F^\times} f(t) \chi_{s + \frac{2n+1}{2}}(t) dt
$$

where

$$
\beta_\psi(\chi_s) = \gamma(s - \frac{2n - 1}{2}, \chi, \psi) \prod_{r=1}^{n} \gamma(2s - 2n + 2r, \chi^2, \psi).
$$
Abelian harmonic analysis

Theorem (JLZ)

For $k > 0$, let $1_k$ be the normalized characteristic function of $1 + \omega^k o_F$, then

$$
\lim_{k \to \infty} \mathcal{L}(1_k)(x)
$$

is stably convergent, i.e. for fixed $x \in F^\times$, there exists $N$ such that $\mathcal{L}(1_k)(x) = \mathcal{L}(1_N)(x)$ for any $k > N$;
Theorem (JLZ)

For $k > 0$, let $1_k$ be the normalized characteristic function of $1 + \varpi^k o_F$, then

$$\lim_{k \to \infty} \mathcal{L}(1_k)(x)$$

is stably convergent, i.e. for fixed $x \in F^\times$, there exists $N$ such that $\mathcal{L}(1_k)(x) = \mathcal{L}(1_N)(x)$ for any $k > N$;

Define

$$\eta_{pvs, \psi}(x) = |x|^{-\frac{2n+1}{2}} \lim_{k \to \infty} \mathcal{L}(1_k)(x).$$

Then $\eta_{pvs, \psi}(x)$ is locally constant on $F^\times$. 
Abelian harmonic analysis

Theorem (JLZ)

The generalized Fourier transform

\[ \mathcal{L} = \mathcal{L}_{\eta_{\text{pvs}}, \psi} : S^+_\text{pvs}(F^\times) \to S^-_{\text{pvs}}(F^\times) \] is given by the following principal value integral

\[ \mathcal{L}(f) = (\eta_{\text{pvs}, \psi} \cdot |\cdot|^\frac{2n+1}{2} * f^\vee), \quad f \in S^+_\text{pvs}(F^\times). \]
Theorem (JLZ)

The generalized Fourier transform
\[ \mathcal{L} = \mathcal{L}_{\eta_{\text{pvs}}, \psi} : S_{\text{pvs}}^+(F^\times) \to S_{\text{pvs}}^-(F^\times) \]
is given by the following principal value integral
\[ \mathcal{L}(f) = (\eta_{\text{pvs}, \psi} \cdot | \cdot |^{\frac{2n+1}{2}} \ast f^\vee), \quad f \in S_{\text{pvs}}^+(F^\times). \]

For any character \( \chi_s = \chi \cdot | \cdot |^s \) of \( F^\times \), the following principal value integral is convergent whenever \( \Re(s) \) is sufficiently small, and admits meromorphic continuation to \( s \in \mathbb{C} \),
\[ \eta_{\text{pvs}, \psi}(\chi_s) := \eta_{\text{pvs}, \psi} \ast \chi_s(e) \]
\[ = \lim_{k \to \infty} \int_{|x| \leq q^k} \eta_{\text{pvs}, \psi}(x) \chi_s(x^{-1}) \, dx \]
\[ = \beta_{\psi}(\chi_s). \]
In conclusion, we develop a new type of harmonic analysis on $F^\times$ associated to $(S_{\text{pvs}}^\pm(F^\times), \mathcal{L}_{\eta_{\text{pvs}}, \psi}, \beta_{\psi}(\chi_s))$. It can be viewed as the abelian case of the Braverman-Kazhdan proposal. This abelian harmonic analysis plays the key role in our work.
Abelian harmonic analysis

- In conclusion, we develop a new type of harmonic analysis on $F^\times$ associated to $(\mathcal{S}^\pm_{pvs}(F^\times), \mathcal{L}_{\eta_{pvs,\psi}, \beta_{\psi}(\chi_s)})$.
- It can be viewed as the abelian case of the Braverman-Kazhdan proposal.
In conclusion, we develop a new type of harmonic analysis on $F^\times$ associated to $(\mathcal{S}_{pvs}^\pm(F^\times), \mathcal{L}_{\eta_{pvs}, \psi}, \beta_\psi(\chi_s))$. It can be viewed as the abelian case of the Braverman-Kazhdan proposal. This abelian harmonic analysis plays the key role in our work.
Harmonic analysis on $M^{ab} \mapsto X_P$

Fix $f \in C_c^\infty(X_P(F))$. Define

$$R_X(f)(g) := \int_{N(F)} f(wng)dn.$$ 

Proposition (JLZ)

The function in $a \in F^\times$

$$F_g(a) := |a|^{(2n+1)}R_X(f)(s(a)g)$$

lies in $S^+_{pvs}(F^\times)$. 
Harmonic analysis on $M^{ab \hookrightarrow X_P}$

Fix $f \in C_c^\infty(X_P(F))$. Define

$$R_X(f)(g) := \int_{N(F)} f(wng)dn.$$ 

**Proposition (JLZ)**

- The function in $a \in F^\times$

$$F_g(a) := |a|^{2n+1} R_X(f)(s(a)g)$$

lies in $S_{pvs}^+(F^\times)$.

- $\mathcal{L}_{\eta_{pvs, \psi}}(F_g)(a) = |a|^{2n+1} \mathcal{F}_{X, \psi}(f)(s^{-1}(a)g)$ lies in $S_{pvs}^-(F^\times)$.
Compatibility between $\mathcal{F}_{\chi,\psi}$ and the unnormalized intertwining operator $M_w(s, \chi)$

Proposition (JLZ)

Let $\mathcal{P}_{\chi_s} : C_c^\infty(X_P(F)) \to I(s, \chi) = \text{Ind}_P^{\text{Sp}_{4n}}(\chi_s)$,

$$\mathcal{P}_{\chi_s}(f)(g) = \int_{F^\times} \chi_s(a)|a|^{\frac{2n+1}{2}} f(s^{-1}(a)g)da.$$ 

Then $\mathcal{P}_{\chi_s}^{-1} \circ \mathcal{F}_{\chi,\psi}(f)(g)$ is absolutely convergent for $\text{Re}(s)$ sufficiently small, and the following identity holds after meromorphic continuation

$$\mathcal{P}_{\chi_s}^{-1} \circ \mathcal{F}_{\chi,\psi}(f)(g) = \beta_\psi(\chi_s)(M_w(s, \chi) \circ \mathcal{P}_{\chi_s})(f)(g).$$
Basic properties of $\mathcal{F}_{X,\psi}$ and $S_{pvs}(X_P(F))$

Define

$$S_{pvs}(X_P(F)) = \mathcal{C}_c^\infty(X_P(F)) + \mathcal{F}_{X,\psi}(\mathcal{C}_c^\infty(X_P(F))).$$

Proposition (JLZ)

$\blacktriangleright$ $\mathcal{F}_{X,\psi}$ stabilizes $S_{pvs}(X_P(F))$. 

Via $P\chi_s$, $S_{pvs}(X_P(F))$ projects onto the space of good sections $I^\dagger(s,\chi)$ introduced by S. Yamana.
Basic properties of $\mathcal{F}_X,\psi$ and $S_{pvs}(X_P(F))$

Define

$$S_{pvs}(X_P(F)) = C_c^\infty(X_P(F)) + \mathcal{F}_X,\psi(C_c^\infty(X_P(F))).$$

Proposition (JLZ)

- $\mathcal{F}_X,\psi$ stabilizes $S_{pvs}(X_P(F))$.
- $|2|^{n(2n+1)} \cdot \mathcal{F}_X,\psi$ extends to a unitary operator on $L^2(X_P(F))$ and $\mathcal{F}_X,\psi \circ \mathcal{F}_X,\psi^{-1} = |2|^{-2n(2n+1)} \text{Id.}$
Basic properties of $\mathcal{F}_{X,\psi}$ and $S_{\text{pvs}}(X_P(F))$

Define

$$S_{\text{pvs}}(X_P(F)) = C^\infty_c(X_P(F)) + \mathcal{F}_{X,\psi}(C^\infty_c(X_P(F))).$$

**Proposition (JLZ)**

- $\mathcal{F}_{X,\psi}$ stabilizes $S_{\text{pvs}}(X_P(F))$.
- $|2|^{n(2n+1)} \cdot \mathcal{F}_{X,\psi}$ extends to a unitary operator on $L^2(X_P(F))$ and $\mathcal{F}_{X,\psi} \circ \mathcal{F}_{X,\psi}^{-1} = |2|^{-2n(2n+1)} \text{Id}$.
- Via $\mathcal{P}_{\chi_s}$, $S_{\text{pvs}}(X_P(F))$ projects onto the space of good sections $\Gamma^+(s,\chi)$ introduced by S. Yamana.
Asymptotic of $S_{pvs}(X_P(F))$

Proposition (JLZ)

A function $f \in C^\infty(X_P(F))$ belongs to $S_{pvs}(X_P(F))$ if and only if $f$ is right $K_{Sp_{4n}}$-finite, and as a function in $a \in F^\times$,

$$|a|^{2n+1} f(s_a^{-1} k)$$

belongs to $S^-_{pvs}(F^\times)$ for any fixed $k \in K_{Sp_{4n}}$.

Therefore functions in $S_{pvs}(X_P(F))$ can be described by their asymptotic behavior near the singular point.
Asymptotic of $S_{\text{pvs}}(X_P(F))$

**Proposition (JLZ)**

A function $f \in \mathcal{C}^\infty(X_P(F))$ belongs to $S_{\text{pvs}}(X_P(F))$ if and only if $f$ is right $K_{\text{Sp}4n}$-finite, and as a function in $a \in F^\times$, 

$$|a|^{2n+1} f(s_a^{-1} k)$$

belongs to $S_{\text{pvs}}^-(F^\times)$ for any fixed $k \in K_{\text{Sp}4n}$.

Therefore functions in $S_{\text{pvs}}(X_P(F))$ can be described by their asymptotic behavior near the singular point.

The support of functions in $S_{\text{pvs}}(X_P(F))$ in $\overline{X}^\text{aff}_P(F)$ is compact. In particular $\overline{X}^\text{aff}_P(F) \setminus X_P(F) = \{\vec{0}\}$. 
Harmonic analysis on $\mathbb{G}_m \times \text{Sp}_{2n} \hookrightarrow X_P$

![Diagram](image)

**Proposition (JLZ)**

- $\mathcal{C} : wN \to \text{Sp}_{2n} \times \{I_{2n}\}$ is given by the Cayley transform.
Harmonic analysis on $\mathbb{G}_m \times \text{Sp}_{2n} \hookrightarrow X_P$

$$\begin{align*}
\text{Sp}_{4n} & \quad \downarrow \quad \text{Sp}_{4n} \\
\downarrow & \\
M^{ab} wN & \quad \rightarrow \quad X_P \quad \leftarrow \quad M^{ab}(\text{Sp}_{2n} \times \{I_{2n}\}) \simeq \mathbb{G}_m \times \text{Sp}_{2n}
\end{align*}$$

Proposition (JLZ)

- $\mathcal{C} : wN \to \text{Sp}_{2n} \times \{I_{2n}\}$ is given by the Cayley transform.
- The Jacobian of $\mathcal{C}^{-1}$ is given by

$$j_{\mathcal{C}^{-1}}(h) = c_0 |\det(h - I_{2n})|^{-(2n+1)}$$

where $c_0 = \frac{1}{\prod_{i=1}^{n} \zeta_F(2i)}$. 
Harmonic analysis on $\mathbb{G}_m \times \text{Sp}_{2n} \hookrightarrow X_P$

For $f \in S_{\text{pvs}}(X_P(F))$, define

$$\phi_f(a, h) := f(s(a)^{-1} \cdot (h, I_{2n}))|a|^{\frac{2n+1}{2}}.$$ 

Set

$$S_{\rho}(G(F)) := \{\phi_f | f \in S_{\text{pvs}}(X_P(F))\}.$$
Harmonic analysis on $\mathbb{G}_m \times \text{Sp}_{2n} \hookrightarrow X_P$

- For $f \in S_{\text{pvs}}(X_P(F))$, define
  \[ \phi_f(a, h) := f(s(a)^{-1} \cdot (h, I_{2n}))|a|^{\frac{2n+1}{2}}. \]

  Set
  \[ S_\rho(G(F)) := \{ \phi_f \mid f \in S_{\text{pvs}}(X_P(F)) \}. \]

- Define
  \[ \Phi_{\rho,\psi}(a, h) := c_0 \cdot \eta_{\text{pvs},\psi}(a \cdot \det(h + I_{2n})) \cdot |\det(h + I_{2n})|^{-\frac{2n+1}{2}}. \]

For $f \in \mathcal{C}_c^\infty(X_P(F))$, the $\rho$-Fourier transform is defined by

\[ \mathcal{F}_{\rho,\psi}(\phi_f)(a, h) := \int_{F^\times} \int_{\text{Sp}_{2n}(F)} \Phi_{\rho,\psi}(ax, gh)\phi_f(x, g)dxdg. \]
Compatibility between $ℱ_{X,\psi}$ and $ℱ_{\rho,\psi}$

Proposition (JLZ)

For $f \in C_{c}^{\infty}(X_{p}(F))$, 

$$
\phi_{ℱ_{X,\psi}}(f)(a, h) = |2|^{-n(2n+1)}ℱ_{\rho,\psi}(\phi_{f})(2^{-2n}a, -h^{-1}).
$$
Compatibility between $\mathcal{F}_{X,\psi}$ and $\mathcal{F}_{\rho,\psi}$

**Proposition (JLZ)**

- For $f \in \mathcal{C}_c^\infty(X_P(F))$,

  $$\phi_{\mathcal{F}_{X,\psi}}(f)(a, h) = |2|^{-n(2n+1)}\mathcal{F}_{\rho,\psi}(\phi_f)(2^{-2n}a, -h^{-1}).$$

- In particular, we can extend the definition of $\mathcal{F}_{\rho,\psi}$ to $S_\rho(G(F))$ via

  $$\phi_{\mathcal{F}_{X,\psi}}(f)(a, h) = |2|^{-n(2n+1)}\mathcal{F}_{\rho,\psi}(\phi_f)(2^{-2n}a, -h^{-1}).$$
Compatibility between $\mathcal{F}_{\rho,\psi}$ and the normalized intertwining operators $M_w^+(s, \chi, \psi)$

**Proposition (JLZ)**

For $h \in \text{Sp}_{2n}(F)$ and $f \in S_{\text{pvs}}(\mathcal{X}_P(F))$,

$$
\mathcal{P}_{\chi s^{-1}} \circ f_{\mathcal{F}_{\rho,\psi}}(\phi_f)((-h^{-1}, \text{Id}_{2n}))
$$

is well-defined for $\Re(s)$ sufficiently small, and the following identity holds after meromorphic continuation to $s \in \mathbb{C}$,

$$
M_w^+(s, \chi, \psi) \circ \mathcal{P}_{\chi s}(f)((h, \text{I})) = \mathcal{P}_{\chi s^{-1}} \circ f_{\mathcal{F}_{\rho,\psi}}(\phi_f)((-h^{-1}, \text{I})).
$$
Basic properties of $S_\rho(G(F))$ and $\mathcal{F}_{\rho,\psi}$

Proposition (JLZ)

- $\mathcal{F}_{\rho,\psi}$ stabilizes $S_\rho(G(F))$. 
Proposition (JLZ)

- $\mathcal{F}_{\rho,\psi}$ stabilizes $S_\rho(G(F))$.
- $\mathcal{F}_{\rho,\psi}$ extends to a unitary operator on $L^2(G(F), dg)$. 
Basic properties of $S_\rho(G(F))$ and $\mathcal{F}_{\rho,\psi}$

Proposition (JLZ)

- $\mathcal{F}_{\rho,\psi}$ stabilizes $S_\rho(G(F))$.
- $\mathcal{F}_{\rho,\psi}$ extends to a unitary operator on $L^2(G(F), dg)$.
- $\mathcal{F}_{\rho,\psi}^{-1} \circ \mathcal{F}_{\rho,\psi} = \text{Id}$. 

Basic properties of $S_\rho(G(F))$ and $\mathcal{F}_{\rho,\psi}$

Proposition (JLZ)

Fix $\chi \otimes \pi \in \text{Irr}(G(F))$. Set

$$Z(s,f,\varphi) = \int_{F^\times \times \text{Sp}_{2n}(F)} \phi(a,h)\varphi(a,h)|a|^{s-\frac{1}{2}} \, d\alpha d\varphi,$$

with $\phi \in S_\rho(G(F)), \varphi \in \mathcal{C}(\chi \otimes \pi)$.

The integral is absolutely convergent for $\text{Re}(s)$ large, and represents a rational function in $q^{-s}$.

- It can be deduced from the asymptotic of functions in $S_{pvs}(X_P(F))$. 

▶ It can be deduced from the asymptotic of functions in $S_{pvs}(X_P(F))$. 

Basic properties of $\mathcal{S}_\rho(G(F))$ and $\mathcal{F}_{\rho,\psi}$

Proposition (JLZ)

- After restriction, the linear functional $\mathcal{Z}(s, \cdot, \cdot)$ lies in

$$\text{Hom}_{G(F) \times G(F)}(C_c^\infty(G(F)) \otimes (\chi_{s-\frac{1}{2}} \otimes \pi^\vee) \otimes (\chi_{s-\frac{1}{2}} \otimes \pi), \mathbb{C}),$$

where the latter space is of dimension 1.
Basic properties of $S_{\rho}(G(F))$ and $\mathcal{F}_{\rho,\psi}$

Proposition (JLZ)

- After restriction, the linear functional $\mathcal{Z}(s, \cdot, \cdot)$ lies in
  $$\text{Hom}_{G(F) \times G(F)}(C_c^\infty(G(F)) \otimes (\chi_{s-\frac{1}{2}} \otimes \pi^\vee) \otimes (\chi_{s-\frac{1}{2}} \otimes \pi), \mathbb{C}),$$
  where the latter space is of dimension 1.

- By equivariant property there exists a rational function $\Gamma_{\rho,\psi}(s, \chi \otimes \pi)$ in $q^{-s}$ such that
  $$\mathcal{Z}(1 - s, \mathcal{F}_{\rho,\psi}(f), \varphi^\vee) = \Gamma_{\rho,\psi}(s, \chi \otimes \pi) \cdot \mathcal{Z}(s, f, \varphi).$$
Basic properties of $S_\rho(G(F))$ and $F_{\rho,\psi}$

Proposition (JLZ)

Let $\varphi_{\chi_s \otimes \pi} \in C(\chi_s \otimes \pi)$. Then as distributions on $G(F)$, the following identity holds by meromorphic continuation,

$$F_{\rho,\psi}(\varphi_{\chi_s \otimes \pi}^\vee) = \Gamma_{\rho,\psi}(\frac{1}{2}, \chi_s \otimes \pi) \cdot \varphi_{\chi_s \otimes \pi}.$$ 

where for $f \in C^\infty_c(G(F))$,

$$(F_{\rho,\psi}(\varphi_{\chi_s \otimes \pi}^\vee), f)_G := (\varphi_{\chi_s \otimes \pi}^\vee, F_{\rho,\psi}(f))_G$$

whenever the latter does not touch the poles. 

In particular $\Gamma_{\rho,\psi}(s, \chi \otimes \pi)$ is a Gamma function in the sense of Gelfand and Graev.
Basic properties of $S_\rho(\mathbb{G}(F))$ and $\mathcal{F}_{\rho,\psi}$

Proposition (JLZ)

Let $\varphi_{\chi_s \otimes \pi} \in \mathcal{C}(\chi_s \otimes \pi)$. Then as distributions on $\mathbb{G}(F)$, the following identity holds by meromorphic continuation,

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In particular $\Gamma_{\rho,\psi}(s, \chi \otimes \pi)$ is a Gamma function in the sense of Gelfand and Graev.

$$
\Gamma_{\rho,\psi}(\frac{1}{2}, \chi_s \otimes \pi) \cdot \Gamma_{\rho,\psi}^{-1}(\frac{1}{2}, \chi_s^{-1} \otimes \pi^\vee) = 1.
$$
Basic properties of $\Phi_{\rho,\psi}$

Set $G_\ell = \{(a, h) \in G(F) = F^\times \times \text{Sp}_{2n}| \ |a| = q^{-\ell}\}$. Let $\text{ch}_\ell$ be the characteristic function of $G_\ell$.

Set $\Phi_{\rho,\psi,\ell} = \Phi_{\rho,\psi} \cdot \text{ch}_\ell$. 
Basic properties of $\Phi_{\rho,\psi}$

Theorem (JLZ)

- The distribution $\Phi_{\rho,\psi,\ell}$ lies in the Bernstein center of $G(F)$. For $\chi \otimes \pi \in \text{Irr}(G(F))$, set

\[ (\chi \otimes \pi)(\Phi_{\rho,\psi,\ell}) = f_{\ell}(\chi \otimes \pi)\text{Id}_{\chi \otimes \pi}. \]
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- The following identity holds after meromorphic continuation
  \[ \sum_{\ell} f_{\ell}(\chi_s \otimes \pi) = \Gamma_{\rho,\psi}\left(\frac{1}{2}, \chi_s^{-1} \otimes \pi^\vee\right) \]
Corollary (JLZ)

Based on the work of Yamana, for any $\chi \otimes \pi \in \text{Irr}(G(F))$, the following set

$$\mathcal{I}_{\chi \otimes \pi} = \{Z(s, \phi, \varphi) | \phi \in S_\rho(G(F)), \varphi \in \mathcal{C}(\chi \otimes \pi)\}$$

is a finitely generated fractional ideal of $\mathbb{C}[q^{-s}, q^s]$ with generator $L(s, \chi \otimes \pi, \rho)$. 
Corollary (JLZ)

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- Based on the work of Lapid-Rallis, Ikeda and Kakuhama, $\Gamma_{\rho, \psi}(s, \chi \otimes \pi) = \gamma(s, \chi \otimes \pi, \rho, \psi)$. 

Verification
Thank you!