Unipotent HC bimodules

jt. w. Mason-Brown & Matvieievskyi.

1) A motivation: G s/simple alg. gr'p$/\mathbb{C}$. One expects a conn'n (Orbit method):

$$\begin{array}{ccc}
\text{Unitary } G\text{-irreps} & & \text{coadj't } G\text{-orbits} \\
\sim\updownarrow & \dashleftarrow\dashrightarrow & \text{(or their equiv't covers)} \\
\{\text{unitarizable simple} & & \cup \\
\text{HC bimodules}\} & & \\
\cup & & \\
?\text{-unipotent bimodules} & \longleftrightarrow & \text{nilp't orbits (\& their covers)}
\end{array}$$

2) Barbasch-Vogan constr'n of "special" unipotent bimodules (1985) (& ideals)

Fact:

$$\begin{array}{ccc}
\{\text{max. 2-sided ideals in } \mathcal{U}(\mathfrak{g})\} & \longleftrightarrow & \mathfrak{h}^*/W \\
\cup\!\!\!\mid & & \in \\
I & \xmapsto{\text{central char'r}} & \overbrace{I \cap Z(\mathcal{U}(\mathfrak{g}))}
\end{array}$$

$Z(\mathcal{U}(\mathfrak{g})) = \mathbb{C}[\mathfrak{h}^*]^W$

Barbasch-Vogan: collection of pts in $\mathfrak{h}^*/W$ ($\leftrightarrow$ max. ideals, I)

Def: Special unip. bimodule = irred'le HC bimodule over $\mathcal{U}(\mathfrak{g})/I$.

$\mathfrak{g}^\vee \supset \mathbb{O}^\vee$ nilp. orbit, $e^\vee \in \mathbb{O}^\vee \rightsquigarrow \mathfrak{sl}_2$-triple $(e^\vee, h^\vee, f^\vee)$

$Gh^\vee$ is uniq. det'd by $\mathbb{O}^\vee$

$\rightsquigarrow$ image of $\frac{1}{2}h^\vee$ in $\mathfrak{g}^\vee /\!/ G^\vee \xrightarrow{\sim} \mathfrak{h}^*/W$

Consider max. ideal $I_{\mathbb{O}^\vee}$ w. central char. $\frac{1}{2}h^\vee$.

Examples: • $\mathbb{O}^\vee = \{0\} \Rightarrow h^\vee = 0 = \frac{1}{2}h^\vee \Rightarrow I_{\mathbb{O}^\vee} = \mathcal{U}(\mathfrak{g})\mathfrak{m}_0$

$$= \mathrm{Ann}_{\mathcal{U}(\mathfrak{g})}(\Delta(-\rho)).$$

• $\mathbb{O}^\vee = \mathbb{O}^\vee_{pr} \Rightarrow h^\vee = 2\rho \Rightarrow \frac{1}{2}h^\vee = \rho \Rightarrow I_{\mathbb{O}^\vee} = \mathcal{U}(\mathfrak{g})\mathfrak{g}.$

BV duality: 2-sided ideal $I \subset \mathcal{U}(\mathfrak{g}) \rightsquigarrow \mathrm{gr}\, I \subset S(\mathfrak{g})$

$\rightsquigarrow$ var'y of $\mathbb{O}$'s $V(\mathrm{gr}\, I) \subset \mathfrak{g}^*$

Fact: If I is maximal (or primitive) $\Rightarrow V(\mathrm{gr}\, I) = \overline{\text{nilp. orbit}}$

Def'n: For $I = I_{\mathbb{O}^\vee}$, this nilp. orbit in $\mathfrak{g}$ is denoted by $d(\mathbb{O}^\vee)$

Example: 1) $\mathfrak{g} = \mathfrak{sl}_n = \mathfrak{g}^\vee$ for $\mathbb{O}^\vee \longleftrightarrow$ part'n $\mu \Rightarrow d(\mathbb{O}^\vee) \leftrightarrow \mu^t$.

2) $\mathfrak{g} = \mathfrak{sp}_4 = \mathfrak{g}^\vee$.

$$\begin{tikzcd}[row sep=small]
\mathbb{O}^\vee \arrow[rr, phantom, "d"] & & \mathbb{O} \\
\{0\} \arrow[rr] & & \mathbb{O}_{pr} \\
\wedge & & \vee \\
\mathbb{O}_{min} \arrow[rr] & & \mathbb{O}_{spr} \\
\wedge & & \vee \\
\mathbb{O}_{spr} \arrow[rruu] & & \mathbb{O}_{min} \notin \mathrm{im}\, d \\
\wedge & & \vee \\
\mathbb{O}_{pr} \arrow[rr] & & \{0\}
\end{tikzcd}$$

In general: $\mathrm{im}\, d = \{\text{special orbits}\}$

So: not all unipotent bimodules are special unipotent.

3) Canonical quant'ns:

$\mathbb{O} \subset \mathfrak{g}^*$ nilp. orbit & $\tilde{\mathbb{O}}$ is G-equiv't cover of $\mathbb{O}$ (if $\mathbb{O} = G/H$ then $\tilde{\mathbb{O}} = G/\underline{H}$ w. $H^\circ \subset \underline{H} \subset H$).

Set $A := \mathbb{C}[\tilde{\mathbb{O}}]$

(i) A is graded ($\mathbb{C}^\times \curvearrowright \mathfrak{g}^*$ by $t.x = t^{-d}x$ lifts to $\tilde{\mathbb{O}}$) & Poisson ($\tilde{\mathbb{O}}$ is sympl'c w. form lifted from $\mathbb{O}$)

compatibility: $\deg\{\cdot,\cdot\} = -d$.

(ii) A is fin. gen'd $\rightsquigarrow X = \operatorname{Spec} A$ (normal affine var'y) & X has symplectic sing's (in sense of Beauville)

(i)+(ii): X is conical sympl'c sing'y.

Thx to (i) can talk about filt'd quant'ns on A

Thx to (ii) have nice classif'n:

Thm (I.L. 16) $\exists$ fin. dim. vector space $\mathfrak{h}_X$ & fin. gr'p $\tilde{W}_X$ w. linear $\tilde{W}_X \curvearrowright \mathfrak{h}_X$ s.t. $\exists$ nat'l bij'n

$$\{\text{filt. quant'ns of } A\} \xleftrightarrow{\sim} \mathfrak{h}_X / \tilde{W}_X.$$

Ex: $Y = T^*(G/P)$ ($P \subset G$ is parabolic), open G-orbit $\tilde{\mathbb{O}} \subset T^*(G/P)$ $\rightsquigarrow A = \mathbb{C}[\tilde{\mathbb{O}}], X$; $\mathfrak{h}_X = (\mathfrak{p}/[\mathfrak{p},\mathfrak{p}])^* = H^2(Y, \mathbb{C}) \ni \lambda$

$\rightsquigarrow$ TDO $\mathcal{D}_{G/P}^{\lambda + \frac{1}{2}c_1(K_{G/P})}$ $\rightsquigarrow$ quant'n $\mathcal{A}_\lambda = \Gamma(\mathcal{D}_{G/P}^{\lambda + \frac{1}{2}c_1(K_{G/P})})$.

Def'n: The canon. quant'n of A is $\mathcal{A}_0$

4) Unipotent ideals & bimodules:

$G \curvearrowright A = \mathbb{C}[\tilde{\mathbb{O}}]$ & comoment map $S(\mathfrak{g}) \longrightarrow A$ deform to $G \curvearrowright \mathcal{A}$ (any quant'n), $U(\mathfrak{g}) \longrightarrow \mathcal{A}$.

Def'n: Unipotent ideal assoc'd to $\tilde{\mathbb{O}}$

$I(\tilde{\mathbb{O}}) = \ker[U(\mathfrak{g}) \longrightarrow \mathcal{A}_0]$ canonical.

Almost Thm (L-M-B-M): Can compute central characters of $I(\tilde{\mathbb{O}})$ is most cases. Using standard comb's, can conclude $I(\tilde{\mathbb{O}})$ is max'l $\Longleftrightarrow \mathcal{A}_0$ is simple.

Conj: $\mathcal{A}_0$ is simple $\forall$ conic. sympl. sing'y.

Sometimes $I(\tilde{\mathbb{O}}_1) = I(\tilde{\mathbb{O}}_2)$ for $\tilde{\mathbb{O}}_1 \neq \tilde{\mathbb{O}}_2$

We can describe corresp. equiv. rel'n on covers geometrically.

Thm (L-M-B-M) Each equiv. class contains a unique max'l element, $\tilde{\mathbb{O}}_{max} \rightsquigarrow$ canon. quant. $\mathcal{A}_0$, $\Gamma := \mathrm{Aut}_G(\tilde{\mathbb{O}}_{max})$ (fin. gr'p)

Then $\Gamma \curvearrowright \mathcal{A}_0$ by filt. alg. autom. & $U(\mathfrak{g})/I(\tilde{\mathbb{O}}_{max}) = \mathcal{A}_0^{\Gamma}$.

Def: A unip. bimodule is an irred. HC bimodule over $U(\mathfrak{g})/I(\tilde{\mathbb{O}})$.

Ex: $\mathfrak{g} = \mathfrak{sl}_2$: $\tilde{\mathbb{O}} = \mathbb{C}^2 \setminus \{0\}$, $\mathcal{A}$ = Weyl algebra $W(\mathbb{C}^2)$,
$\Gamma = \{\pm 1\} \rightsquigarrow W(\mathbb{C}^2)^\Gamma = U(\mathfrak{g})/I$, $I \leftrightarrow$ h. wt $-\frac{1}{2}$
(Canon. quant. of princ. orbit $\leftrightarrow -1$)
Isotyp. comp's of $W(\mathbb{C}^2)$ are 2 unip. bimodules.

Thm: $\tilde{\mathbb{O}} = \tilde{\mathbb{O}}_{max}$. Then

$$\begin{tikzcd} \{\Gamma\text{-irreps}\} \arrow[rr, leftrightarrow, "\sim"] & & \{\text{unip. } U(\mathfrak{g})/I(\tilde{\mathbb{O}})\text{-bimodules}\} \\ \tau \arrow[u, phantom, "\rotatebox{90}{$\in$}"] \arrow[rr, mapsto] & & \mathrm{Hom}_\Gamma(\tau, \mathcal{A}_0) \arrow[u, phantom, "\rotatebox{90}{$\in$}"] \end{tikzcd}$$

5)
$$\begin{tikzcd} \text{Special unip} \arrow[r, Rightarrow] & \text{unip} \\ \mathbb{O}^\vee \arrow[u] \arrow[r, dashed] & \tilde{\mathbb{O}}/\sim \arrow[u] \end{tikzcd}$$

Thm: $\exists$ inj've map $\tilde{d}: \{\mathbb{O}^\vee\} \longrightarrow \{\tilde{\mathbb{O}}/\sim\}$
s.t. $\tilde{d}(\mathbb{O}^\vee)$ is a cover of $d(\mathbb{O}^\vee)$ & $I_{\mathbb{O}^\vee} = I(\tilde{d}(\mathbb{O}^\vee))$

$\mathbb{O}^\vee \rightsquigarrow$ Slodowy slice $S^\vee \rightsquigarrow X^\vee := S^\vee \cap \mathcal{N}^\vee$ - sing'r sympl'c var'y

Symplectic duality

Speculation: from a sing'r sympl'c var'y $(X^\vee)$ - with some "decoration" one should be able to construct its "symp'l dual" X w. certain properties
Take $X = \operatorname{Spec} \mathbb{C}[\tilde{d}(\mathbb{O}^\vee)]$

Case 1: $\mathbb{O}^\vee$ is distinguished ($\Longleftrightarrow e^\vee \notin$ proper Levi)

"$\Rightarrow$" X is "rigid" (has no Poisson deform'ns)

$\tilde{\mathbb{d}}(\mathbb{O}^\vee) :=$ univ. cover of $\mathbb{d}(\mathbb{O})$

Case 2: $\mathbb{O}^\vee = G\mathbb{O}_L^\vee$ ($\mathbb{O}_L^\vee \subset \mathfrak{l}^\vee$-nilp. orbit) $\Longrightarrow$

$\tilde{\mathbb{d}}(\mathbb{O}^\vee) :=$ open orbit in $G \times^P (\tilde{\mathbb{O}}_L \times \mathfrak{n})$.