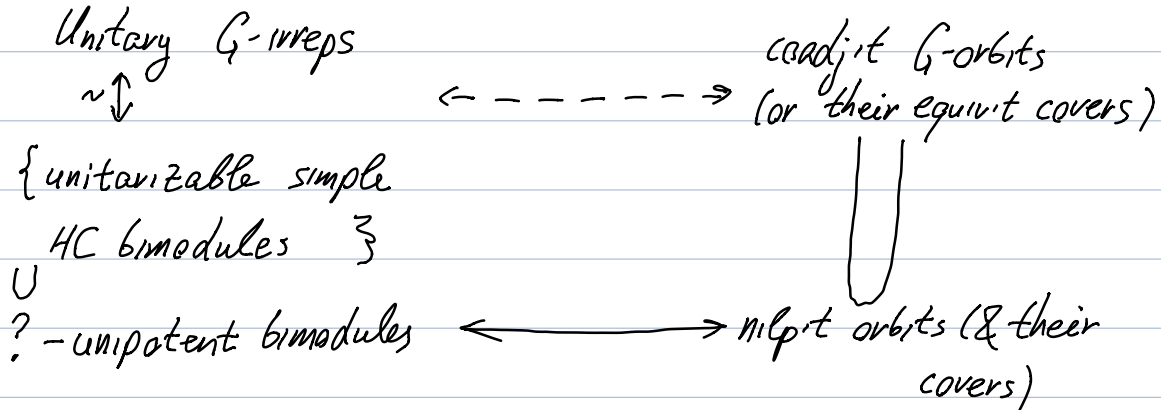


Unipotent HC bimodules

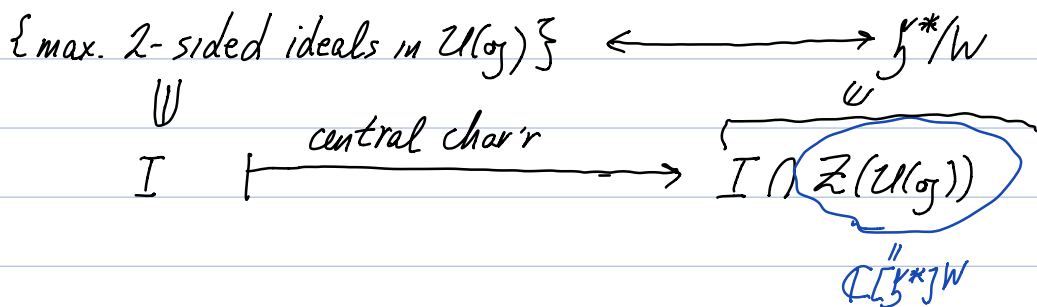
jt. w. Mason-Brown & Matvieievskiy.

1) A motivation: G s/simple alg. grp/ \mathbb{C} . One expects a conn'n (Orbit method):



2) Barbasch-Vogan constr'n of "special" unipotent bimodules (1985) (& ideals)

Fact:



Barbasch-Vogan: collection of pts in \mathfrak{g}^*/W (\leftrightarrow max. ideals, \mathcal{I})

Def: Special unip. bimodule = irred'le HC bimodule over $\mathcal{U}(\mathfrak{g})/I$.

\mathcal{O}^v is unig. det'd by \mathcal{O}^v

1] $\mathfrak{g}^v \supset \mathcal{O}^v$ nilp. orbit, $e^v \in \mathcal{O}^v \rightsquigarrow \mathfrak{sl}_2$ -triple $(e^v, \overline{h^v}, f^v)$

\leadsto image of $\frac{1}{2}h^\vee$ in $\mathfrak{g}^\vee // G^\vee \xrightarrow{\sim} \mathfrak{h}^*/W$

Consider max. ideal $I_{\mathcal{O}^\vee}$ w. central char. $\frac{1}{2}h^\vee$.

Examples: $\bullet \mathcal{O}^\vee = \{0\} \Rightarrow h^\vee = 0 = \frac{1}{2}h^\vee \Rightarrow I_{\mathcal{O}^\vee} = \mathcal{U}(\mathfrak{g})\mathfrak{h}_0$
 $= \text{Ann}_{\mathcal{U}(\mathfrak{g})}(\Delta(-\rho)).$

$\bullet \mathcal{O}^\vee = \mathcal{O}_{pr}^\vee \Rightarrow h^\vee = 2\rho \Rightarrow \frac{1}{2}h^\vee = \rho \Rightarrow I_{\mathcal{O}^\vee} = \mathcal{U}(\mathfrak{g})\mathfrak{g}.$

BV duality: 2-sided ideal $I \subset \mathcal{U}(\mathfrak{g}) \leadsto \text{gr } I \subset S(\mathfrak{g})$

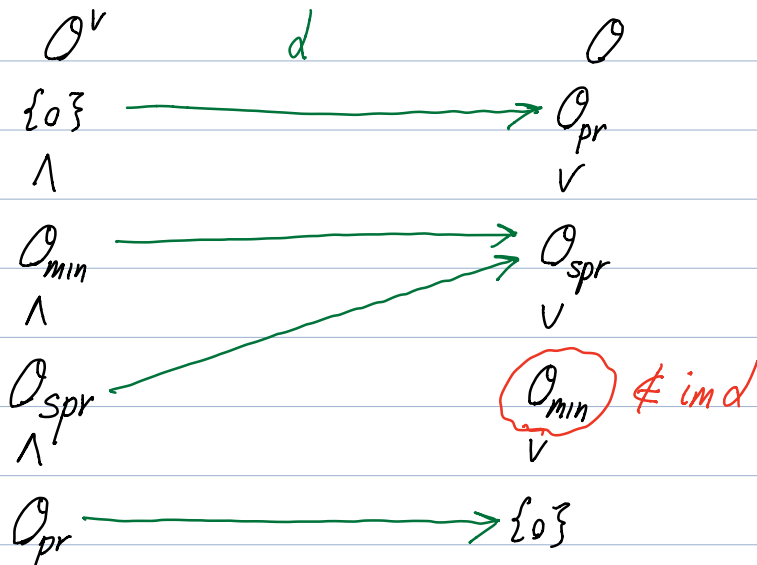
\leadsto var'y of \mathcal{O} 's $V(\text{gr } I) \subset \mathfrak{g}^*$

Fact: If I is maximal (or primitive) $\Rightarrow V(\text{gr } I) = \overline{\text{nilp. orbit}}$

Def'n: For $I = I_{\mathcal{O}^\vee}$, this nilp. orbit in \mathfrak{g} is denoted by $d(\mathcal{O}^\vee)$

Example: 1) $\mathfrak{g} = \mathfrak{sl}_2 = \mathfrak{g}^\vee$ for $\mathcal{O}^\vee \leftrightarrow \text{part'n } \mu \Rightarrow d(\mathcal{O}^\vee) \leftrightarrow \mu^\natural$

2) $\mathfrak{g} = \mathfrak{sp}_4 = \mathfrak{g}^\vee$



In general: $\text{im } d = \{\text{special orbits}\}$

2]

So: not all unipotent bimodules are special unipotent.

3) Canonical quantization:

$\mathcal{O} \subset \mathfrak{g}^*$ nilp. orbit & $\tilde{\mathcal{O}}$ is G -equiv. cover of \mathcal{O} (if $\mathcal{O} = G/H$ then $\tilde{\mathcal{O}} = G/\underline{H}$ w. $H^0 \subset \underline{H} \subset H$).

Set $A := \mathbb{C}[\tilde{\mathcal{O}}]$

(i) A is graded ($\mathbb{C}^x \curvearrowright \mathfrak{g}^*$ by $t \cdot x = t^{-d} x$ lifts to $\tilde{\mathcal{O}}$) & Poisson ($\tilde{\mathcal{O}}$ is symplectic w. form lifted from \mathcal{O})

compatibility: $\deg \{ \cdot, \cdot \} = -d$.

(ii) A is fin. gen'd $\leadsto X = \text{Spec } A$ (normal affine var'y) & X has symplectic sing's (in sense of Beauville)

(i)+(ii): X is conical symplectic sing'y.

Thx to (i) can talk about filt'd quant'ns on A

Thx to (ii) have nice classif'n:

Thm (I.L. 16) \exists fin. dim. vector space \mathfrak{h}_X & fin. grp \tilde{W}_X w. linear $\tilde{W}_X \curvearrowright \mathfrak{h}_X$ s.t. \exists nat'l bij'n

$$\{\text{filt. quant'ns of } A\} \xleftrightarrow{\sim} \mathfrak{h}_X / \tilde{W}_X.$$

Ex: $Y = T^*(G/P)$ ($P \subset G$ is parabolic), open G -orbit $\tilde{\mathcal{O}} \subset T^*(G/P)$
 $\leadsto A = \mathbb{C}[\tilde{\mathcal{O}}]$, X ; $\mathfrak{h}_X = (\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}])^* = H^2(Y, \mathbb{C}) \ni \lambda$

\leadsto TDO

$\mathcal{D}_{G/P}^{\lambda + \frac{1}{2}c_1(K_{G/P})}$

\leadsto quant'n $\mathcal{A}_\lambda = \Gamma(\mathcal{D}_{G/P}^{\lambda + \frac{1}{2}c_1(K_{G/P})})$.

Def'n: The canon. quant'n of A is \mathcal{A}_0

4) Unipotent ideals & bimodules:

$G \curvearrowright A = \mathbb{C}[\tilde{\mathcal{O}}]$ & comoment map $S(\mathfrak{g}) \rightarrow A$ deform to
 $G \curvearrowright \mathcal{A}$ (any quant'n), $U(\mathfrak{g}) \rightarrow \mathcal{A}$.

Def'n: Unipotent ideal assoc'd to $\tilde{\mathcal{O}}$

$$I(\tilde{\mathcal{O}}) = \ker [U(\mathfrak{g}) \rightarrow \mathcal{A}_0] \xleftarrow{\text{canonical.}}$$

Almost Thm (L-M-B-M): Can compute central characters of $I(\tilde{\mathcal{O}})$ in most cases. Using standard comb's, can conclude $I(\tilde{\mathcal{O}})$ is max'l $\Leftrightarrow \mathcal{A}_0$ is simple.

Conj: \mathcal{A}_0 is simple \nleftrightarrow conic. simpl. sing'y.

Sometimes $I(\tilde{\mathcal{O}}_1) = I(\tilde{\mathcal{O}}_2)$ for $\tilde{\mathcal{O}}_1 \neq \tilde{\mathcal{O}}_2$

We can describe corresp. equiv. rel'n on covers geometrically.

Thm (L-M-B-M) Each equiv. class contains a unique max'l element,

$\tilde{\mathcal{O}}_{\max} \rightsquigarrow$ canon. quant. \mathcal{A}_0 , $\Gamma := \text{Aut}_G(\tilde{\mathcal{O}}_{\max})$ (fin. gr'p)

Then $\Gamma \curvearrowright \mathcal{A}_0$ by f.l.t. alg. autom. & $U(\mathfrak{g})/I(\tilde{\mathcal{O}}_{\max}) = \mathcal{A}_0^\Gamma$.

Def: A unip. bimodule is an irred. HC bimodule over $U(\mathfrak{g})/I(\tilde{\mathcal{O}})$.

Ex: $\mathfrak{g} = \mathfrak{sl}_2$: $\tilde{\mathcal{O}} = \mathbb{C}^2 \setminus \{0\}$, $\mathcal{A} =$ Weyl algebra $W(\mathbb{C}^2)$,
 $\Gamma = \{\pm 1\} \rightsquigarrow W(\mathbb{C}^2)^\Gamma = \mathcal{U}(\mathfrak{g})/I$, $I \leftrightarrow$ h. wt $-\frac{1}{2}$
 (Canon. quant. of princ. orbit $\leftrightarrow -1$)
 Isotyp. comp's of $W(\mathbb{C}^2)$ are 2 unip. bimodules.

Thm: $\tilde{\mathcal{O}} = \tilde{\mathcal{O}}_{\max}$. Then

$$\begin{array}{ccc} \{\Gamma\text{-irreps}\} & \xleftrightarrow{\sim} & \{\text{unip. } \mathcal{U}(\mathfrak{g})/I(\tilde{\mathcal{O}})\text{-bimodules}\} \\ \cup & & \cup \\ \tau & \longmapsto & \text{Hom}_\Gamma(\tau, \mathcal{A}_0) \end{array}$$

5) Special unip \Rightarrow unip

$$\begin{array}{ccc} \uparrow & & \uparrow \\ \mathcal{O}^\vee & \dashrightarrow & \tilde{\mathcal{O}}/\sim \end{array}$$

Thm: \exists inj've map $\tilde{d}: \{\mathcal{O}^\vee\} \longrightarrow \{\tilde{\mathcal{O}}/\sim\}$
 s.t. $\tilde{d}(\mathcal{O}^\vee)$ is a cover of $d(\mathcal{O}^\vee)$ & $I_{\mathcal{O}^\vee} = I(\tilde{d}(\mathcal{O}^\vee))$

$\mathcal{O}^\vee \rightsquigarrow$ Slodowy slice $S^\vee \rightsquigarrow X^\vee := S^\vee \cap N^\vee$ - sing'r symplectic variety

Symplectic duality.

Speculation: from a sing'r symplectic variety (X^\vee) - with some "decoration" one should be able to construct its "symp'l dual" X w. certain properties

Take $X = \text{Spec } \mathbb{C}[\tilde{d}(\mathcal{O}^\vee)]$

Case 1: \mathcal{O}^\vee is distinguished ($\Leftrightarrow e^\vee \notin$ proper Levi)

" \Rightarrow " X is "rigid" (has no Poisson deformations)

$\tilde{d}(\mathcal{O}^\vee) :=$ univ. cover of $d(\mathcal{O})$

Case 2: $\mathcal{O}^\vee = G \mathcal{O}_L^\vee$ ($\mathcal{O}_L^\vee \subset \mathcal{L}^\vee$ -mlp. orbit) \Rightarrow

$\tilde{d}(\mathcal{O}^\vee) :=$ open orbit in $G \times^P (\tilde{\mathcal{O}}_L \times \mathcal{K})$.