

From families in Weyl groups to unipotent elements.

G. Lusztig

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G almost simple simply connected / k alg closed char $p \neq 0$
 p not bad prime

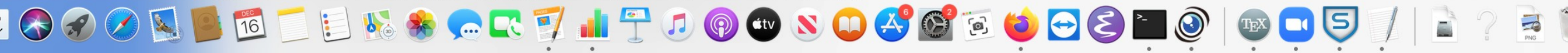
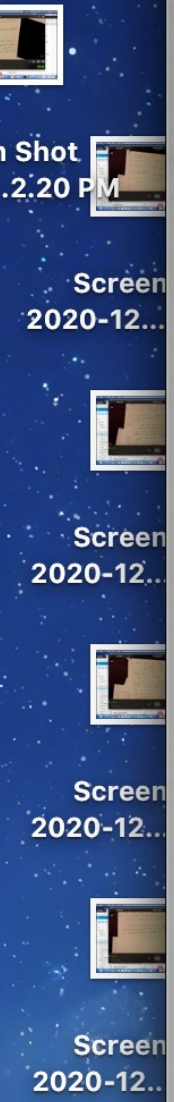
W Weyl group, $un(G)$ set of unip. classes [S. 1976]
 $Irr W = \bigsqcup_{C \in un(G)} Irr W_C$: Springer's partition

$\gamma : Irr W \rightarrow \mathbb{N}$
 $\gamma(E) = \dim(\text{Springer fibre at } u \in C, \text{ where } E \in Irr W_C)$

Another partition of $Irr W$ [L. 1979] "families".
 $a \neq m$ generic degree polynomial attached to E is $\frac{1}{mE} q^{a(E)} + \text{higher}$

$a : Irr W \rightarrow \mathbb{N}$
 $m : Irr W \rightarrow \mathbb{N}_{>0}$
 a is constant on families

Claim. Springer's partition and function γ can be re constructed purely in terms of families and a -function



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(W_{af}, S_{af}) : affine Weyl group associated to G . ^{finite} K (a Weyl gr.)
 For $K \subsetneq S_{af}$, $W_K = \text{subgp. of } W_{af} \text{ gener. by } K$
 can be identified with a subgroup of $W_{af}/\text{transl.} = W$.

For $E \in \text{Irr } W$ let $\Gamma(E) = \{ (K, E') ; K \subsetneq S_{af}, E' \in \text{Irr } W_K, (E', E|_{W_K}) \neq 0 \}$

Let $c_E = \max (a_{E'} ; (K, E') \in \Gamma(E))$, $a_{E'}$ def. in terms of W_K
 $\Gamma_M(E) = \{ (K, E') \in \Gamma(E) ; a_{E'} = c_E \}$ if E_1, E_2 in $\text{Irr } W$ we say $E_1 \sim E_2$ if $(K_1, E'_1) \in \Gamma_M(E_1), (K_2, E'_2) \in \Gamma_M(E_2)$
 E'_1, E'_2 in same family of $W_{K_i} = W_{K_2}$.
 s.t. $K_1 = K_2$, $c : \text{Irr } W \rightarrow \mathbb{N}$ constant on c -families.

Equivalence classes: c -families = pieces of Springer's partition
 claim: (1) c -families = pieces of Springer's partition
 (2) $c_E = \chi(E) \quad \forall E \in \text{Irr } W$.

Unipotent blocks.

$$Ls(G) = \left\{ (c, \mathcal{L}) \mid \begin{array}{l} c \in \text{un}(G) \\ \mathcal{L} \text{ inv. loc. sys.} \\ \text{on } C \end{array} \right\}$$

Springer (1976) $\text{Irr } W \leftrightarrow Ls(G)$ a subset of $Ls(G)$

$$[L-1984] \quad Ls(G) = \coprod \text{unip. blocks, one of which is } Ls(G)$$

Each unip. block \mathfrak{B} is in can-bijection with $\text{Irr } W_{\mathfrak{B}}$
 $W_{\mathfrak{B}}$ a certain Weyl group.

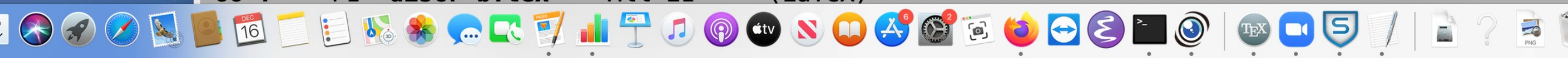
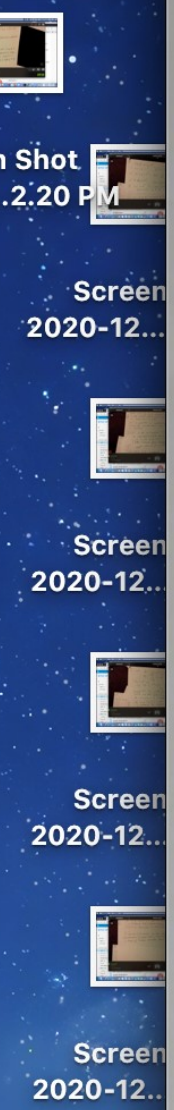
claim: The unip. blocks and the Weyl groups $W_{\mathfrak{B}}$ can be reconstructed purely in terms of W_{af} .

Let $\Omega = \text{group of autom. of } (W_{\text{af}}, S_{\text{af}})$ such that the induced autom of $W_{\text{af/transl.}}$ is inner.

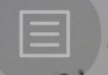
For any $w \in \Omega$ we can define a set $\mathcal{E}_w(W)$ of subsets $K \subsetneq S_{\text{af}}$ with $w(K) = K$ such that

$$\{ \text{unip. blocks} \} \leftrightarrow \coprod_{w \in \Omega} \mathcal{E}_w(W)$$

The Weyl group $W_{\mathfrak{B}}$ can be defined in terms of $\mathcal{E}_w(W)$.
 the corresp. $K \in \mathcal{E}_w(W)$.



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One property of ~~the~~ K is that a Chevalley group / F_2 with Weyl group W_K and Froh. action defined by w should admit a unip. exp. repres.

We can formulate this condition without reference to groups over F_2 . \Rightarrow "sharp Weyl groups".

Let (W, S') be an irred. Weyl group

For $E \in \text{Irr } W$, let $z(E)$ largest integer s.t.

$(q+1)^{z(E)}$ | generic degree polyn. attached to E .

We have $z(E) \leq \#$ (orbits of oppos. on S') = $n(\text{opp})$.

Def. W' is sharp if $\left\{ \begin{array}{l} \exists E_0 \in \text{Irr } W', z(E_0) = n(\text{opp}) \\ n(\text{opp}) \text{ odd (opp)} = \text{even (excludes } E_7) \end{array} \right.$

If $a: W \rightarrow W'$ is an autom preserving S' but not reversing \Rightarrow or \boxtimes

we say that (W', a) is sharp if W' is sharp and $\text{ord}(\text{opp} \cdot a) = \text{odd}$

List

$\{1\}$

${}^2A_{\frac{t-1}{8}-1}$

$t = 5, 7, 9, \dots$

$B_{\frac{t-1}{4}}$

$t = 3, 5, 7, \dots$

$D_{\frac{t}{4}}$

$t = 4, 8, 12, \dots$

${}^2D_{\frac{t}{4}}$

$t = 6, 10, 14, \dots$

$G_2, {}^3D_4, F_4, {}^2E_6, E_8$

Screenshot

6) Can form 2 graphs

I.

$$1 - \frac{B^{\frac{3-1}{4}}}{D^{\frac{4^2}{4}}} - \frac{B^{\frac{5-1}{4}}}{2D^{\frac{6^2}{4}}} - \frac{B^{\frac{7-1}{4}}}{D^{\frac{8^2}{4}}} - \dots - \frac{B^{\frac{9-1}{4}}}{D^{\frac{10^2}{4}}}$$

II

$$1 \times 1 - \frac{{}^2A^{\frac{5-1}{8}}}{\left(\frac{B_{(6/2)^2-1}}{\frac{4}{4}}\right)} - \frac{{}^3A^{\frac{7-1}{8}}}{\left(\frac{D_{(6/2)^2-1} \times B_{(6/2)^2-1}}{\frac{4}{4}}\right)} - \dots - \frac{{}^2A^{\frac{9-1}{8}}}{\left(\frac{B_{(10/2)^2-1} \times B_{(10/2)^2-1}}{\frac{4}{4}}\right)}$$

$$\frac{{}^2A^{\frac{11-1}{8}}}{\dots}$$

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Screenshot

7 For $w \in \mathcal{J}$ or $K \neq \emptyset$ and $C_w(W)$ is the set of all $K \notin \text{Sep}$ so that either $K = \emptyset$ or $K \neq \emptyset$ and

- (1) K is stable under any $w' \in \mathcal{J}$
- (2) W_K is sharp (rel. to w)
- (3) $W_{\text{Sep}} - K$ is w -irreducible
- (4) If $W_{\mathcal{A}}$ is of type B then W_K is the product of two joined vertices in 1st graph
- (5) If $W_{\mathcal{A}}$ is of type C or D and w has $\ll 1$ fixed point on Sep then W_K is the product of two joined vertices in 2nd graph.

Example

