From families in Weyl groups to unipotent elements.

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G almost simple simply connected $/k$ alg closed char $p \geq 0$, p not bad prime

W Weyl group, $un(G)$ set of unip. classes

$\mathrm{Irr}\, W = \coprod_{C \in un(G)} \mathrm{Irr}\, W_C$: Springer's partition [S. 1976]

$\gamma : \mathrm{Irr}\, W \to \mathbf{N}$

$\gamma(E) = \dim(\text{Springer fibre at } u \in C, \text{ where } E \in \mathrm{Irr}\, W_C)$

Another partition of $\mathrm{Irr}\, W$ [L. 1979] "families".

$a : \mathrm{Irr}\, W \to \mathbf{N}$ generic degree polynomial attached to E is

$m : \mathrm{Irr}\, W \to \mathbf{N}_{>0}$ $\quad \frac{1}{m_E} q^{a(E)} + \text{higher}$

a is constant on families

<u>Claim</u>. Springer's partition and function γ can be reconstructed purely in terms of families and a-function

(W_{af}, S_{af}) : affine Weyl group associated to G^{dual}.

For $K \subsetneq S_{af}$, W_K = subgp. of W_{af} gener. by K (a finite Weyl gr.) can be identified with a subgroup of $W_{af}/\text{transl.} = W$.

For $E \in \operatorname{Irr} W$ let

$$\Gamma(E) = \{ (K, E') ; \; K \subsetneq S_{af}, \; E' \in \operatorname{Irr} W_K, \; (E' : E|_{W_K}) \neq 0 \}$$

Let $c_E = \max (a_{E'} ; (K, E') \in \Gamma(E))$, $a_{E'}$ def. in terms of W_K

$$\Gamma_M(E) = \{ (K, E') \in \Gamma(E) ; \; a_{E'} = c_E \}.$$

For E_1, E_2 in $\operatorname{Irr} W$ we say $E_1 \sim E_2$ if

$\exists \; (K_1, E_1') \in \Gamma_M(E_1), \; (K_2, E_2') \in \Gamma_M(E_2)$

s.t. $K_1 = K_2$, E_1', E_2' in same family of $W_{K_1} = W_{K_2}$.

Equivalence classes: c-families. $\quad$ $c : \operatorname{Irr} W \to \mathbb{N}$ constant on c-families.

__claim__ : (1) c-families = pieces of Springer's partition

(2) $c_E = \gamma(E) \quad \forall \; E \in \operatorname{Irr} W$.

3)

Example: G_2 springer partition

Irr W	1_0	2_1	1_3	2_2	$\tilde{1}_3$	1_6
γ_E	0	1	1	2	3	6
Irr W_K, $K=G_2$ families	1_0	2_1	1_3	2_2	$\tilde{1}_3$	1_6
a	0	1	1	1	1	6
Irr W_K $K=A_1\times A_1$	1_0	1_1		1_2	$\tilde{1}_1$	
Irr W_K, $K=A_2$	1_0	2_1			1_3	

Unipotent Blocks.

$\ell s(G) = \left\{ (c, \mathcal{L}) \;\middle|\; \begin{array}{l} c \in \mathrm{un}(G) \\ \mathcal{L} \text{ irr. loc. sys. on } c \end{array} \right\}$

Springer (1976) $\quad \mathrm{Irr}\, W \leftrightarrow {}'\ell s(G)$ a subset of $\ell s(G)$

[L. 1984] $\quad \ell s(G) = \coprod$ unip. blocks, one of which is ${}'\ell s(G)$

Each unip. block β is in can-bijection with $\mathrm{Irr}\, W_\beta$

W_β a certain Weyl group.

<u>claim</u> : The unip. blocks and the Weyl groups W_β can be reconstructed purely in terms of W_{af}.

Let Ω = group of autom. of (W_{af}, S_{af}) such that the induced autom of $W_{af}/\text{transl.}$ is inner.

For any $\omega \in \Omega$ we can define a set $\mathcal{E}_\omega(W)$ of subsets $K \subsetneqq S_{af}$ with $\omega(K) = K$ such that

$$\{\text{unip. blocks}\} \leftrightarrow \coprod_{\omega \in \Omega} \mathcal{E}_\omega(W)$$

The Weyl group W_β can be defined in terms of [illegible]. the corresp. $K \in \mathcal{E}_\omega(W)$.

One property of ~~?~~ K is that a Chevalley group $/ F_q$ with Weyl group W_K and Frob. action defined by ω should admit a unip. cusp. repres.

We can formulate this condition without reference to groups over F_q. $\Rightarrow$ "sharp Weyl groups".

Let (W', S') be an irred. Weyl group

For $E \in \mathrm{Irr}\, W$, let $z(E)$ largest integer s.t. $(q+1)^{z(E)} \mid$ generic degree polyn. attached to E.

We have $z(E) \leq \#(\text{orbits of oppos. on } S') = r(\text{opp})$.

<u>Def.</u> W' is <u>sharp</u> if $\begin{cases} \exists\, E_0 \in \mathrm{Irr}\, W', \quad z(E_0) = r(\text{opp}) \\ r(\text{opp})\, \mathrm{ord}(\text{opp}) = \text{even} \end{cases}$ (excludes E_7)

If $a: W' \to W'$ is an autom preserving S' but not reversing $\Rightarrow$ or $\Leftarrow$ we say that (W', a) is sharp if W' is sharp and $\mathrm{ord}(\text{opp} \cdot a) = \text{odd}$

<u>List</u>

$\{1\}$

${}^2A_{\frac{t^2-1}{8}-1}$ $\quad t = 5, 7, 9, \ldots$

$B_{\frac{t^2-1}{4}}$ $\quad t = 3, 5, 7, \ldots$

$D_{\frac{t^2}{4}}$ $\quad t = 4, 8, 12, \ldots$

${}^2D_{\frac{t^2}{4}}$ $\quad t = 6, 10, 14, \ldots$

$G_2, {}^3D_4, F_4, {}^2E_6, E_8$

6)

Can form 2 graphs

I.

$$
\begin{tikzcd}
1 \arrow[r, dash] & B_{\frac{3^2-1}{4}} \arrow[r, dash] & D_{\frac{4^2}{4}} \arrow[r, dash] & B_{\frac{5^2-1}{4}} \arrow[r, dash] & {}^2D_{\frac{6^2}{4}} \arrow[r, dash] & B_{\frac{7^2-1}{4}} \arrow[d, dash] \\
 & & & \cdots \arrow[r, dash] & B_{\frac{9^2-1}{4}} \arrow[r, dash] & D_{\frac{8^2}{4}}
\end{tikzcd}
$$

II.

$$
\begin{tikzcd}
1\times 1 \arrow[r, dash] & {}^2A_{\frac{5^2-1}{8}-1} \arrow[r, dash] & {}^a\left(B_{\frac{(6/2)^2-1}{4}} \times B_{\frac{(6/2)^2-1}{4}}\right) \arrow[r, dash] & {}^2A_{\frac{7^2-1}{8}-1} \arrow[d, dash] \\
{}^a\left(B_{\frac{(10/2)^2-1}{4}} \times B_{\frac{(10/2)^2-1}{4}}\right) \arrow[d, dash] & {}^2A_{\frac{9^2-1}{8}-1} \arrow[l, dash] & & {}^a\left(D_{\frac{(8/2)^2}{4}} \times D_{\frac{(8/2)^2}{4}}\right) \arrow[ll, dash] \\
{}^2A_{\frac{11^2-1}{8}-1} \arrow[r, dash] & \cdots & &
\end{tikzcd}
$$

7

For $\omega \in \Omega$ $C_\omega(W)$ is the set of all $K \subsetneqq S_{af}$ so that either $K = \emptyset$ or $K \neq \emptyset$ and

(1) K is stable under any $\omega' \in \Omega$

(2) W_K is sharp (rel. to ω)

(3) $W_{S_{af} - K}$ is ω-irreducible

(4) If W_{af} is of type B then W_K is the product of two joined vertices in 1^{st} graph

(5) If W_{af} is of type C or D and ω has ≤ 1 fixed point on S_{af} then W_K is the product of two joined vertices in 2^{nd} graph.

<u>Example</u>

W is

K is $\emptyset$

or

or