Macdonald Polynomials and counting parabolic bundles

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.

Plan

1) Introduction to counting

2) Symmetric functions

3) Main result

4) Applications

Introduction

Ways to obtain interesting q-functions:

1) Rep theory: say trace(some intertwiner)

2) Coherent algebraic geometry: Say χ(some vector bundle)

3) Count points over finite fields. ("= constructible algebraic geometry")

Remark Sometimes these counts can be explained as dimensions of cohomology groups. Example

$$|\mathbb{P}^n(\mathbb{F}_q)| = 1+q+\dots+q^n$$

but in many applications the information flows from $|X(\mathbb{F}_q)|$ to $H^*(X)$, and not the other way around! (see Harder-Narasimhan)

Symmetric functions (The way to package and process information obtained from the previous slide)

Level 0 $(X = (x_1, x_2, \dots)$

$$p_k = \sum x_i^k \qquad e_k = \sum_{i_1 < \dots < i_k} x_{i_1} x_{i_2} \dots x_{i_k} \qquad h_k = \sum_{i_1 \le \dots \le i_k} x_{i_1} x_{i_2} \dots x_{i_k}$$

plethysms e.g.

$$f(p_1, p_2, p_3, \dots)\left[\frac{X}{1-q}\right] = f\left(\frac{\sum x_i}{1-q}, \frac{\sum x_i^2}{1-q^2}, \frac{\sum x_i^3}{1-q^3}, \dots\right)$$

Cauchy kernel

$$\mathrm{Exp}[XY] = \exp\left(\sum_n \frac{1}{n} p_n(X) p_n(Y)\right)$$

(see λ-rings)

scalar product $(p_\lambda, p_\mu) = \delta_{\lambda\mu} z_\lambda \qquad (h_\lambda, m_\mu) = \delta_{\lambda\mu}$

$p_\lambda := \prod p_{\lambda_i}, \quad h_\lambda = \prod h_{\lambda_i} \qquad m_\lambda = \sum$ distinct permutations of $x_1^{\lambda_1} x_2^{\lambda_2} \dots$

Symmetric functions (cont.)

Level 1 Schur functions $\displaystyle s_\lambda = \sum_{\substack{\text{SSYT}\\ T}} x_T$

$$\begin{array}{|ccc} 1 & 1 & 3 \\ 2 & 2 & \\ 3 & 4 & \end{array}$$

SSYT

rep theory:

ρ_λ S_n-rep $\displaystyle s_\lambda = \frac{1}{n!} \sum_{\pi \in S_n} \operatorname{Tr} \rho_\lambda(\pi)\, p_{\text{cycle type}(\pi)}$

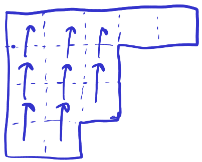
or: $\operatorname{Tr} \rho_\lambda(\pi) = \left(s_\lambda, p_{\text{cycle type}(\pi)} \right)$

(These are not so obvious from the viewpoint of Level 0)

Level 2 Hall-Littlewood functions

$\mathbb{C} := \mathbb{F}_q$

N_λ = nilpotent matrix of type λ

λ 

$\dim \operatorname{Ker} N_\lambda = \lambda_1$

$\dim \operatorname{Ker} N_\lambda^2 / N_\lambda = \lambda_2$

$\dots$

e.g. $N_{(n)} = 0$ N_{1^n} = single Jordan block

How many partial flags of type μ are preserved by N_λ?

$$0 = F_0 \subset F_1 \subset \dots \subset F_{\ell(\mu)} = \mathbb{C}^n$$

$\dim F_1/F_0 = \mu_1$

$\dim F_2/F_1 = \mu_2$

$\dots$

Answer: $K_{\mu\lambda}(q)$

$$H_\lambda(X,q) := \sum_\mu K_{\mu\lambda}(q)\, m_\mu$$

Examples

$H_{1^n} = \sum_{|\mu|=n} m_\mu = h_n$ $H_{(n)} = \sum_{|\mu|=n} m_\mu \cdot |\mathrm{Flag}_\mu(\mathbb{F}_q)|$

$n=2$: $H_{1^2} = h_2$ $H_{(2)} = m_2 + (q+1) m_{1,1} = h_2 + q e_2$

Macdonald polynomials

Definition (2 upper-triangularities + normalization)

$\widetilde{H}_\lambda[X;q,t]$ (usually omit q,t)

$\widetilde{H}_\lambda[(t-1)X] \in \mathrm{span}(m_\mu \mid \mu \leq \lambda)$

$\widetilde{H}_\lambda[(q-1)X] \in \mathrm{span}(m_\mu \mid \mu \leq \lambda')$

$\widetilde{H}_\lambda[1] = 1$

Example

$\widetilde{H}_{(n)}[X] = H_{(n)}[X]$

(our H-L friend)

$\widetilde{H}_{1^n}[X] = H_{(n)}[X;t]$

$q \longleftrightarrow t$

$\lambda \longleftrightarrow \lambda'$

symmetry!

What is known?

traces of intertwiners ✓ (Etingof-Kirillov)

χ(vector bundle) ✓ (Haiman)

(sort of)

constructible ? (Hausel-Letellier-Rodriguez-Villegas conjecture on mixed Hodge polynomials of character varieties of Riemann surfaces gives us a hint)

Main result X smooth complete curve $/\mathbb{F}_q$

We want to count vector bundles on X (usually produces infinite sums)

Schiffmann suggests truncation:

Def $\mathcal{C}$ abelian category. A truncation is a full subcat $\mathcal{C}_- \subset \mathcal{C}$. Objects of $\mathcal{C}_-$ are called good or negative so that:

For SES $0 \to E \to F \to G \to 0$
1) F good $\Rightarrow$ E good
2) E, G good $\Rightarrow$ F good.

Main example $X = \mathbb{P}^1$ then $E = \mathcal{O}(m_1) \oplus \mathcal{O}(m_2) \oplus \dots$
E is good if $m_i \le 0$.

Also fix λ $|\lambda| = n$.

Go over pairs $(E$ good bundle, $\theta \in \mathrm{End}(E))$ nilpotent of type λ at generic point.

Remark This is already interesting. (Schiffmann counted these)

Pick $s_1, \dots, s_k \in X(\mathbb{F}_q)$, we can "probe" each (E, θ) at these s_i, for instance by counting how many flags does θ preserve at s_i. Or by measuring $\mathrm{type}(\theta(s_i))$.

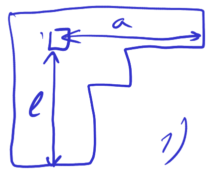
The natural generating function: $X_1 = (x_{11}, x_{12}, \dots)$ $X_2 = (x_{21}, x_{22}, \dots)$ $\dots X_k$

$$\Omega_\lambda = \sum_E \frac{t^{-\deg E}}{|\mathrm{Aut}\, E|} \sum_{\theta \in \mathrm{End}(E)} H_{\mathrm{type}\,\theta(s_1)}[X_1, q]\, H_{\mathrm{type}\,\theta(s_2)}[X_2, q] \dots$$

$\theta \sim N_\lambda$ generically

Theorem $\Omega_\lambda = \Omega_\lambda^{\mathrm{Schiffmann}}(q,t)\, \tilde{H}_\lambda[X_1, q, t]\, \tilde{H}_\lambda[X_2, q, t] \dots \tilde{H}_\lambda[X_k, q, t]$

for $X = \mathbb{P}^1$ $\quad \Omega_\lambda^{\mathrm{Schiffmann}} = \dfrac{1}{\prod_{\square \in \lambda} (q^a - t^{\ell+1})(q^{a+1} - t^\ell)}$



1) normalization: clear.

Proof 1) prove factorization by local analysis. 2) upper triangularity
2) check defining properties of $\tilde{H}_\lambda$; 3) $\mathbb{P}^1$ with $k=2$.

local analysis

Crucial property given $(\mathcal{E}, \theta)$ $\quad \mathcal{E}$ good $\Leftrightarrow$ $\operatorname{Ker}\theta$ good

So we can modify $\mathcal{E}$ at s_i without changing $\operatorname{Ker}\theta$;
in particular we can straighten it, i.e.
make type $\theta(s_i)$ = type generically

M

ways to straighten N_λ
many to many correspondence

Conjugacy classes of nilpotent matrices over $\mathbb{F}_q[z]$

$$\longrightarrow \sum_{\substack{\text{conj. classes}\\ \text{of nilp.}\\ \text{matrices}/\\ \mathbb{F}_q[z]}} \frac{|M/G_1|}{|G_2\backslash M|} t^{\operatorname{ord} g}_{H_{\theta(0)}}$$

$$Z(N_\lambda) \cap G(O) = G_2$$

$$Z(\theta_\lambda) \cap G(O) = G_1$$

$$M = \{ g \in G(\mathbb{F}_q(z)) / g\theta g^{-1} = N_\lambda \}$$

s.t. 1) poles of g are bounded
2) g preserves $\operatorname{Ker}\theta$.

Applications

HLR-V $\to$ Poincaré polynomials of character varieties of Riemann surfaces with semisimple generic monodromies

nabla identities (with Carlsson)

Count bundles $/\mathbb{P}^1$ $\quad \mathcal{E} \to \mathcal{E}(-d)$

$$\mathcal{E} = \mathcal{O}(-m_1) \oplus \cdots \mathcal{O}(m_n) \qquad m_i \geq 0$$

$$\sum_{m_1, \dots, m_n} t^{|m|} \frac{q^{\mathrm{dinv}(m)}}{(q-1)^n \prod [\mu_i]_q !} = \left(\nabla^d h_n\left[\frac{X}{(q-1)(1-t)}\right], h_n \right)$$

and generalizations ...