Hitchin fibration and commuting schemes

Joint work with B. C. Ngô

Plan:

① Introduction to commuting schemes

② Invariant theory for commuting schemes

③ Applications to Hitchin fibration for higher dim varieties

④ Proof of the Chevalley restriction theorem for symplectic Lie algebras

① Commuting Schemes

Fixed $k = \bar{k}$ char $k = 0$. Let G be a reductive gp / k and let $\mathfrak{g} = \mathrm{Lie}\, G$ be its Lie algebra.
For every $d \in \mathbb{Z}_{\geq 0}$, we can consider the following map

$$\mathfrak{g}^d \longrightarrow \prod_{i<j} \mathfrak{g} \qquad (x_1, \dots, x_d) \longrightarrow \prod_{i<j} [x_i, x_j]$$

Ex: $d = 2$ $\qquad \mathfrak{g}^2 = \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g} \qquad (x, y) \longrightarrow [x, y]$

The commuting scheme of $\mathfrak{g}$ denoted by $\mathfrak{C}^d_{\mathfrak{g}}$ is the following fiber product

$$\begin{tikzcd} \mathfrak{C}^d_{\mathfrak{g}} \arrow[r] \arrow[d] & \{0\} \arrow[d] \\ \mathfrak{g}^d \arrow[r] & \prod_{i<j} \mathfrak{g} \end{tikzcd}$$

The k-points of $\mathfrak{C}^d_{\mathfrak{g}}$ consist of $(x_1, \dots, x_d) \in \mathfrak{g}^d$ such that $[x_i, x_j] = 0$ for all $1 \leq i < j \leq d$

Ex: For $d=2$ $\mathfrak{g}=\mathfrak{gl}_n$, then

$$\mathfrak{C}^2_{\mathfrak{gl}_n} = \{(X,Y)\in \mathfrak{gl}_n^2 \mid XY-YX=0\}$$

Very little is known about $\mathfrak{C}^d_{\mathfrak{g}}$!!

- For $d=2$, Richardson proved that $\mathfrak{C}^2_{\mathfrak{g}}$ is irreducible (generalizing an earlier result of Gerstenhaber in the case $\mathfrak{g}=\mathfrak{gl}_n$)
- For $d\geq 3$, $\mathfrak{C}^d_{\mathfrak{g}}$ is in general reducible

Conjecture: $\mathfrak{C}^2_{\mathfrak{g}}$ is normal and reduced.

Still open even for $G=GL_n$ (n large)

Rmks: $\mathfrak{C}^d_{\mathfrak{g}}$, $d\geq 3$ is unlikely to be reduced.

② Invariant theory for $\mathfrak{C}_\mathfrak{g}^d$

$$\begin{array}{ccc} \mathfrak{C}_\mathfrak{g}^d & \hookrightarrow & \mathfrak{g}^d \\ \circlearrowleft & & \circlearrowleft \leftarrow \text{diagonal adjoint action} \\ G & & G \end{array}$$

Let $\mathfrak{C}_\mathfrak{g}^d /\!/ G = \operatorname{spec}\left(k[\mathfrak{C}_\mathfrak{g}^d]^G\right)$

Conjecture: $\mathfrak{C}_\mathfrak{g}^d /\!/ G$ is normal and reduced

Fix a maximal torus $T \subset G$ with Weyl gp $W = N_G(T)/T$ and let $\mathfrak{t} \subset \mathfrak{g}$ be the Cartan subalgebra. Since $[\mathfrak{t}, \mathfrak{t}] = 0$ We have a natural map $\mathfrak{t}^d \longrightarrow \mathfrak{C}_\mathfrak{g}^d$ which gives rise to a map

$$c : \mathfrak{t}^d / W \longrightarrow \mathfrak{C}_\mathfrak{g}^d /\!/ G$$

Conjecture: The map $c: \mathfrak{t}^d/W \longrightarrow \mathfrak{C}^d_{\mathfrak{g}}/\!/G$ is an isomorphism.

Rmks ① Since $\mathfrak{t}^d/W$ is known to be normal and reduced the Conj above $\Rightarrow$ $\mathfrak{C}^d_{\mathfrak{g}}/\!/G$ is normal and reduced

② When $d=1$ $\mathfrak{C}^1_{\mathfrak{g}} = \mathfrak{g}$, $\mathfrak{C}^d_{\mathfrak{g}}/\!/G = \mathfrak{g}/\!/G$ and the Conj is the well-known Chevalley restriction thm: $\mathfrak{t}/W \xrightarrow{\sim} \mathfrak{g}/\!/G$

Thus one can view the Conj as a version of Chevalley restriction thm for commuting schemes.

③ Known cases:

- Gan-Ginzburg: $G = GL_n$ $d=2$
  (used Etingof-Ginzburg's results on SRA)
- Francesco Vaccarino $G = GL_n$ arbitrary d
  (used results of Deligne and Procesi)

Thm (Ngô, –): The map $c : \mathfrak{t}^d/W \to \mathfrak{g}^d/\!/G$ is an isomorphism for $G = Sp_{2n}$

Rmks ① The proof generalizes the one for GL_n due to Vaccarino. The key ingredient is a certain multiplicative property of the Pfaffian.

② The case $G = SO_n$ is work in progress.

③ There is also a "derived" version of the conj.

③ Hitchin fibration for higher dim varieties

Let X be a smooth proj variety of $\dim = d$ $\quad G = GL_n$

<u>$d=1$</u>

$$\begin{array}{ccccccc} G\times\mathbb{G}_m & & & & \mathbb{G}_m & & \mathbb{G}_m \\ \curvearrowright & & & & \curvearrowright & & \curvearrowright \\ \mathfrak{g} & \xrightarrow{\chi} & \mathfrak{g}/\!\!/G & \simeq & \mathfrak{t}/W & \simeq & \prod_{i=1}^{n} \mathbb{A}^1_{(i)} \end{array}$$

$$t\cdot x = t^i x$$

$$[\chi] : \left[\mathfrak{g}/G\times\mathbb{G}_m\right] \longrightarrow \left[(\mathfrak{t}/W)/\mathbb{G}_m\right]$$

The map $[\chi]$ plays an important role in the study of Hitchin fibration for curves.

$$h_X(\mathcal{E},\theta) = (\mathrm{Tr}\,\theta, \cdots, \mathrm{Tr}\,\theta^n)$$

$$\begin{array}{ccc}
\mathrm{Higgs}_X & \xrightarrow{h_X} & \mathcal{A}_X \\
\| & & \| \\
\left\{ (\mathcal{E},\theta) \,\middle|\, \begin{array}{l} \mathcal{E}\in \mathrm{Bun}_n(X) \\ \theta \in H^0(X, \mathrm{End}(\mathcal{E})\otimes\Omega_X) \end{array} \right\} & & \bigoplus_{i=1}^{n} H^0(X, \Omega_X^{i}) \\
\wr\| & & \| \\
\left\{ \begin{tikzcd} X \arrow[r, "\theta"] \arrow[dr, "\Omega_X"'] & {[\mathfrak{g}/G\times\mathbb{G}_m]} \arrow[d] \\ & B\mathbb{G}_m \end{tikzcd} \right\} & \xrightarrow{[\chi]} & \left\{ \begin{tikzcd} X \arrow[r] \arrow[dr, "\Omega_X"'] & {[(\mathfrak{t}/W)/\mathbb{G}_m]} \simeq {[\prod \mathbb{A}^1_{(i)}/\mathbb{G}_m]} \arrow[d] \\ & B\mathbb{G}_m \end{tikzcd} \right\}
\end{array}$$

$d \geq 2$

$$GL_d \circlearrowright \mathfrak{g}^d \simeq \mathrm{Hom}(\mathbb{A}^d, \mathfrak{g}) \circlearrowleft G$$

$$\overset{G\times GL_d}{\circlearrowright}\mathcal{C}^d_{\mathfrak{g}} \xrightarrow{\chi} \mathcal{C}^d_{\mathfrak{g}} /\!/ G \simeq \overset{GL_d}{\circlearrowright}\mathfrak{t}^d/W \xhookrightarrow{\mathrm{pol}_W} \overset{GL_d}{\circlearrowright}\prod_{i=1}^{n} S^i \mathbb{A}^d$$

$$[v_1, \cdots, v_n] \longrightarrow (v_1 + \cdots + v_n,\ v_1v_2 + v_2v_3 + \cdots,\ \cdots,\ v_1v_2\cdots v_n)$$

$$\mathfrak{t}^d/W \simeq (\mathbb{A}^d)^n/S_n$$

$$\begin{tikzcd} \mathrm{Higg}_X \arrow[r, "sd_X"] \arrow[rr, bend left, "h_X"] & B_X \subset & A_X \end{tikzcd}$$

$h_X \leftarrow$ Hitchin-Simpson map

$$\mathrm{Higg}_X = \left\{ (\mathcal{E}, \theta) \;\middle|\; \begin{array}{l} \text{i) } \mathcal{E} \in \mathrm{Bun}_n(X) \\ \text{ii) } \theta : H^0(X, \mathrm{End}(\mathcal{E}) \otimes \Omega_X) \\ \text{iii) } \theta \wedge \theta = 0 \end{array} \right\}$$

Locally, $\theta = \sum_{i=1}^{d} \theta_i \, dx_i \quad \theta_i : X \to \mathfrak{gl}_n$

$\theta \wedge \theta = 0 \iff [\theta_i, \theta_j] = 0$

$$A_X = \bigoplus_{i=1}^{n} H^0(X, S^i \Omega_X)$$

$$\mathrm{Higg}_X \cong \left\{ \begin{tikzcd} X \arrow[r, "\theta"] \arrow[dr, "\Omega_X"'] & {[\mathfrak{C}_{\mathfrak{g}}^d / G \times GL_d]} \arrow[d] \\ & BGL_d \end{tikzcd} \right\} \xrightarrow{[\chi]} \left\{ \begin{tikzcd} X \arrow[r] \arrow[dr, "\Omega_X"'] & {[(\mathfrak{t}^d / W) / GL_d]} \arrow[d] \\ & BGL_d \end{tikzcd} \right\} \xrightarrow{pol_W} \left\{ \begin{tikzcd} X \arrow[r] \arrow[dr] & {[\prod S^i A^d / GL_d]} \arrow[d] \\ & BGL_d \end{tikzcd} \right\}$$

Rmks: ① It follows that the Hitchin-Simpson map in higher-dim is not surjective in general and we expect that $sd_X : \mathrm{Higg}_X \to B_X$ is an abelian fibration

② One can form the universal Cameral cover

$$\begin{tikzcd} \tilde{X} \arrow[r] \arrow[d] & {[\mathfrak{t}^d/GL_d]} \arrow[d] \\ X \times B_X \arrow[r, dashed] & {[(\mathfrak{t}^d/W)/GL_d]} \end{tikzcd}$$

and we expect that the fibers of sd_X admit a description in terms of sheaves on $\tilde{X}$

Both expectations are true in the case $d=2$ $G=GL_2$

---

④ Chevalley restriction thm for GL_n and Sp_{2n}

Thm (Ngo, -) The map $c: \mathfrak{t}^d/W \longrightarrow \mathfrak{C}^d_{\mathfrak{g}}/\!/G$ is an iso for $G = GL_n$ and Sp_{2n}.

Pf: We will construct a map $s: \mathfrak{C}^d_{\mathfrak{g}}/\!/G \longrightarrow \mathfrak{t}^d/W$ and show that $s \circ c$ and $c \circ s$ are identity maps

The construction of S in the case $G = Gl_n$ is due to Deligne: It suffices to construct a Gl_n-equ map $\mathfrak{C}_{g}^{d} \longrightarrow t^d/W$.

Let R be any k-algebra and $x = (x_1, \dots, x_d) \in \mathfrak{C}_{g}^{d}(R)$, $x_i \in g(R) = gl_n(R)$. The R-point x gives rise to a map

$$\theta : k[v_1, \dots, v_d] \longrightarrow gl_n(R).$$
$$v_i \longmapsto x_i$$

The composition $k[v_1, \dots, v_d] \xrightarrow{\theta} gl_n(R) \xrightarrow{\det} R$ is a polynomial map of homogenous degree $n \Longrightarrow$

$$\begin{tikzcd}
f \in k[v_1, \dots, v_d] \arrow[r, "\theta"] \arrow[d] & gl_n(R) \arrow[r, "\det"] & R \\
f^{\otimes n} \in \left(k[v_1, \dots, v_d]^{\otimes n}\right)^{S_n} \arrow[urr, "\exists!\, \alpha"'] & &
\end{tikzcd}$$

such that $\alpha(f^{\otimes n}) = \det(\theta(f))$

Since $\det$ is multiplicative $\Rightarrow \alpha$ is a map of k-algebra and thus gives rise to a R-point

$$\mathrm{spec}(R) \longrightarrow \mathrm{spec}\left(\left(k[v_1 \dots v_d]^{\otimes n}\right)^{S_n}\right) \simeq t^d/W$$

The construction above defines a GL_n-inv map $\mathfrak{C}^d_{\mathfrak{g}} \longrightarrow \mathfrak{t}^d/W$ and hence a map $s: \mathfrak{C}^d_{\mathfrak{g}}//G \longrightarrow \mathfrak{t}^d/W$

To check $s \circ c = c \circ s = id$ we use the result of Procesi:

Thm (Procesi): $k[\mathfrak{gl}_n^d]^{GL_n}$ (and hence $k[\mathfrak{C}^d_{\mathfrak{g}}]^G$) is generated by the functions: $1 \le i_1, \dots, i_N \le d$, $N \in \mathbb{Z}_{\geq 0}$

$$\mathfrak{g}^d \longrightarrow k$$
$$(A_1, \dots, A_d) \longrightarrow Tr(A_{i_1} \cdots A_{i_N})$$

$G = Sp_{2n}$ case

The key step is the construction of the symplectic version of the Deligne map $s: \mathfrak{C}^d_{\mathfrak{g}}//G \longrightarrow \mathfrak{t}^d/W$

Let $J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$ $\quad \theta(g) = J ({}^t g)^{-1} J^{-1}$

$$Sp_{2n} = \{ g \in Gl_{2n} \mid \theta(g) = g \}$$

$$\mathfrak{g} = \mathfrak{sp}_{2n} = \{ g \in \mathfrak{gl}_{2n} \mid {}^t x J = -Jx \}$$

$$\cup$$

$$\mathfrak{t} = \begin{bmatrix} t_1 & & & & & \\ & \ddots & & & & \\ & & t_n & & & \\ & & & -t_1 & & \\ & & & & \ddots & \\ & & & & & -t_n \end{bmatrix} \quad W = S_n \ltimes (\pm 1)^n$$

Cartan decomposition: $\mathfrak{gl}_{2n} = \mathfrak{g} \oplus \mathfrak{g}_1$

where $\mathfrak{g}_1 = \{ g \in \mathfrak{gl}_{2n} \mid {}^t x J = J x \}$

<u>Fact</u>: For $x, y \in \mathfrak{g}$ (or $x, y \in \mathfrak{g}_1$), $xy = yx$

then $xy = yx \in \mathfrak{g}_1$

<u>Fact</u>: $\mathfrak{g}_1 \xrightarrow{\sim} \mathfrak{alt}_{2n} = \{ x \in \mathfrak{gl}_{2n} \mid {}^t x = -x \}$

$y \longrightarrow Jy$

and the Pfaffian $Pf : \mathfrak{alt}_{2n} \longrightarrow k$ defines a polynomial map $N : \mathfrak{g}_1 \xrightarrow{\sim} \mathfrak{alt}_{2n} \xrightarrow{Pf} k$ of homogeneous degree n

<u>Fact</u> (multiplicative property of $N : \mathfrak{g}_1 \to k$)

For $x, y \in \mathfrak{g}_1(R)$ $\quad xy = yx$

$$\Rightarrow \quad N(\underset{\substack{\uparrow \\ \mathfrak{g}_1}}{xy}) = N(x)N(y)$$

Construction of symplectic Deligne map:

Let $x = (x_1, \dots, x_d) \in \mathfrak{L}^d_g(R)$

$$\Rightarrow$$

$$\begin{tikzcd}
k[v_1, \dots, v_d] \arrow[r, "\theta"] & \mathfrak{gl}_{2n}(R) & v_i \mapsto x_i \\
k[v_1 \cdots v_d]_{\text{even}} \arrow[u, phantom, "\cup"] \arrow[r, "\theta_1"] \arrow[dr] & \mathfrak{g}_1(R) \arrow[u, phantom, "\cup"] \arrow[r, "N"] & R \\
f \mapsto f^{\otimes n} & \left(k[v_1 \cdots v_d]^{\otimes n}_{\text{even}}\right)^{S_n} \arrow[ur, "\alpha"'] &
\end{tikzcd}$$

α ↑ k-algebra map

$\Rightarrow$ Get a R-point $\alpha \in \mathfrak{t}^d/W(R)$

$$\alpha : \operatorname{Spec} R \to \operatorname{Spec}\left(\left(k[v_1 \cdots v_d]^{\otimes n}\right)^{S_n \ltimes (\pm 1)^n}\right) = \mathfrak{t}^d/W$$

This finishes the construction of the symplectic Deligne map $\mathfrak{C}_{\mathfrak{g}}^{d} \longrightarrow \mathfrak{t}^{d}/W$ which gives rise to a map $s: \mathfrak{C}_{\mathfrak{g}}^{d}//G \longrightarrow \mathfrak{t}^{d}/W$.

Finally, one use Procesi's result for symplectic gps to show that s is the inverse of $c: \mathfrak{t}^{d}/W \to \mathfrak{C}_{\mathfrak{g}}^{d}/G$ $\square$

<u>Rmk</u>: Let $\mathfrak{C}_{\mathfrak{g}_1}^{2} = \{(x,y) \in \mathfrak{g}_1^{2} \mid [x,y]=0\}$

If we know $\mathfrak{C}_{\mathfrak{g}_1}^{2}$ is reduced $\Rightarrow$ Key Fact

$N(xy)$
$= N(x)N(y)$