Hitchin fibration and commuting schemes

Joint work with B.C. Ngô

Plan:

1. Introduction to commuting schemes
2. Invariant theory for commuting schemes
3. Applications to Hitchin fibration for higher dim varieties
4. Proof of the Chevalley restriction theorem for symplectic Lie algebras
0 Commuting Schemes

Fixed $k=\bar{k}$, char $k=0$. Let $G$ be a reductive group $G/k$ and let $g = \text{Lie} G$ be its Lie algebra. For every $d \geq 2$, we can consider the following map

$$g^d \to \prod_{i=1}^d g \quad (x_{i_1}, \ldots, x_{i_d}) \to \prod_{i \leq j} [x_i, x_j]$$

**Example:** $d=2$, $g^2 = g \times g \to g$ \quad $(x, y) \to [x, y]$

The commuting scheme of $g$ denoted by $C^d_g$ is the following fiber product

$$C^d_g \to \{ 0 \} \quad \downarrow \quad \downarrow$$

$$g^d \to \prod_{i < j} g$$

The $k$-points of $C^d_g$ consist of $(x_{i_1}, \ldots, x_{i_d}) \in g^d$ such that $[x_{i_i}, x_{i_j}] = 0$ for all $1 \leq i < j \leq d$. 
**EX:** For $d=2$, $g=gl_n$, then

$$C_{gl_n}^2 = \{ (X,Y) \in gl_n^2 \mid XY - YX = 0 \}$$

Very little is known about $C_g^d$.

- For $d=2$, Richardson proved that $C_g^2$ is irreducible (generalizing an earlier result of Gerstenhaber in the case $g=gl_n$).
- For $d>3$, $C_g^d$ is in general reducible.

**Conjecture:** $C_g^2$ is normal and reduced.

Still open even for $G=GL_n$ ($n$ large).

**Rmk:** $C_g^d$, $d>3$ is unlikely to be reduced.
Invariant theory for $E^d_g$

$E^d_g \rightarrow \mathfrak{g}^d$

$G \times G \leftarrow$ diagonal adjoint action

Let $E^d_g \text{}//G = \text{spec}(\mathcal{O}[E^d_g]_G)$

Conjecture: $E^d_g \text{}//G$ is normal and reduced

Fix a maximal torus $T \subset G$ with Weyl gp $W = N_G(T)/T$ and let $t \subset g$ be the Cartan subalgebra. Since $[t, t] = 0$ we have a natural map $t^d \rightarrow E^d_g$ which gives rise to a map $c : t^d/W \rightarrow E^d_g//G$
Conjecture: The map $\psi : t^d \to \mathcal{E}^d_{g}/G$ is an isomorphism.

Ranks 1 since $(t^d)_{W}$ is known to be normal and reduced, the Conj above $\Rightarrow \mathcal{E}^d_{g}/G$ is normal and reduced.

(3) When $d = 1$, $\mathcal{E}^1_{g} = G$, $\mathcal{E}^d_{g}/G = \mathfrak{g}/G$.

and the Conj is the well-known Chevalley restriction thm. $\psi : t^d \to \mathfrak{g}/G$.

Thus one can view the Conj as a version of Chevalley restriction thm for commuting schemes.

(3) Known cases:

- Gan-Ginzburg: $G = GL_n$ $d = 2$ (used Etingof-Ginzburg's results on SRA).
- Francesco Vaccarino $G = GL_n$ arbitrary $d$ (used results of Deligne and Procesi).
Thm \((N_{G_0}(-))\): The map \(c : \frac{t^d}{W} \rightarrow G^d/G\) is an isomorphism for \(G = \text{Sp}_{2n}\).

Remarks 1 The proof generalizes the one for \(Gln\) due to Vaccarino. The key ingredient is a certain multiplicative property of the Pfaffian.

2 The case \(G = SO_n\) is work in progress.

3 There is also a "derived" version of the conj.
3. Hitchin fibration for higher \( \dim \) varieties

Let \( X \) be a smooth proj. variety of \( \dim = d \) \( G = G_{\mathbb{C}^n} \)

\[
d = 1 \quad G \times G_{\mathbb{C}^n} \quad \mathbb{G}_m \times \mathbb{G}_m
\]

\[
q \times \frac{q}{G} \sim \frac{q}{W} \sim \prod_{\tilde{c} = 1}^{n} \mathbb{A}^1_{(\tilde{c})}
\]

\[
t \cdot x = t^c x
\]

\[
[x] : \left[ \frac{q}{G \times G_{\mathbb{C}^n}} \right] \to \left[ \frac{\left( \frac{q}{W} \right)}{G_{\mathbb{C}^n}} \right]
\]

The map \( [x] \) plays an important role in the study of Hitchin fibration for curves.
$h_x : (\varepsilon, \theta) \rightarrow \mathfrak{A}^x$

$\xi \rightarrow \left[ \frac{G_x g_m}{} \right] \downarrow \left[ \frac{G_x g_m}{} \right] \downarrow \left[ \frac{G_x g_m}{} \right] \downarrow \left[ \frac{G_x g_m}{} \right]

A \rightarrow 2

G_{ld} \cong g^d \cong \text{Hom}(A^d, g) \cong G

G \times G_{ld}

\bigwedge

E^d_{g} \rightarrow E^d_{g} / G = \frac{E^d}{W} \rightarrow \prod_{i=1}^{n} S^i / A^d

\left[ v_{1, \ldots, v_n} \right] \rightarrow (v_1 + \ldots + v_n, v_1, v_2 + \ldots, v_n)\frac{E^d}{W} \cong \left( A^d \right)^n / S_n$
\( h_x \in \) Hitchin-Simpson map

\[
\begin{aligned}
\text{Higgs}_x & \quad \xrightarrow{sd_x} \quad B_x \subset A_x \\
\{(s, \theta) \mid & \exists \theta_0 \in \text{Bun}_n(X) \} \\
\theta : H^0 (X, \text{End}(E) \otimes \Omega^1_X) & \\
\text{locally, } \theta = \sum_{i=1}^d \theta_i dx_i \quad \theta_i : X \to GL_n \\
\theta \wedge \theta = 0 \Leftrightarrow [\theta_0, \theta_0] = 0
\end{aligned}
\]

\( \text{Rmk: } \) It follows that the Hitchin-Simpson map in higher-dim is not surjective in general and we expect that \( sd_x : \text{Higgs}_x \to B_x \) is an abelian fibration.
(2) One can form the universal Camera cover 
\[ \tilde{X} \rightarrow \left[ \frac{t^d}{G_\ell} \right] \] 
and we expect that the fibers of \( \text{sol}_X \) admit a description in terms of sheaves on \( \tilde{X} \).

Both expectations are true in the case 
\[ d = 2 \quad G_\ell = G_{\ell_2} \]

(4) Chevalley restriction theorem for \( G_{\ell n} \) and \( S_{P_{2n}} \).

Theorem (Ngo, -) The map \( C: \frac{t^d}{W} \rightarrow \frac{G_{\ell n}}{G} \) 
is an iso for \( G_\ell = G_{\ell n} \) and \( S_{P_{2n}} \).

Proof: We will construct a map \( S: \frac{G_{\ell n}}{G} \rightarrow \frac{t^d}{W} \) and show that \( SC \) and \( CS \) are identity maps.
The construction of $S$ in the case $G=GL_n$ is due to Deligne. It suffices to construct a $GL_n$-equivariant map $S^d_{g} \rightarrow \mathbb{P}^{d}$.

Let $R$ be any $\mathbb{R}$-algebra and $x=(x_1, \ldots, x_d) \in S^d_{g}(R)$.

$x_i \in g\mathfrak{l}(R)$, $g\mathfrak{l}_n(R)$. The $R$-point $x$ gives rise to a map $\Theta : R[V_1, \ldots, V_d] \rightarrow g\mathfrak{l}_n(R)$.

$V_i \mapsto x_i$

The composition $R[V_1, \ldots, V_d] \rightarrow g\mathfrak{l}_n(R) \xrightarrow{\text{det}} R$

is a polynomial map of homogeneous degree $\nu \Rightarrow R[V_1, \ldots, V_d] \rightarrow g\mathfrak{l}_n(R) \xrightarrow{\text{det}} R$

\[ f \mapsto e \left( \left( R[V_1, \ldots, V_d] \otimes k \right)^{\otimes n} \right)^{\otimes n+1} \]

such that $\alpha(f \otimes n) = \text{det} (\Theta(f))$

Since $\text{det}$ is multiplicative $\Rightarrow \alpha$ is a map of $\mathbb{R}$-algebra and thus gives rise to a $R$-point $\text{spec}(R) \rightarrow \text{spec} \left( \left( \left( R[V_1, \ldots, V_d] \otimes k \right)^{\otimes n} \right)^{\otimes n+1} \right) \simeq \mathbb{P}^{d}$.
The construction above defines a GL-n inv map \( C^d_g \rightarrow t^d/W \) and hence a map \( s: C^d_g//G \rightarrow t^d/W \).

To check \( s \circ c = c \circ s = c^d \), we use the result of Procesi:

**Theorem (Procesi):** \( \mathfrak{h}[gl^d] \) GLn (and hence \( \mathfrak{h}[C^d_g//G] \)) is generated by the functions:

\[
g^\mathfrak{g}_d \rightarrow \mathfrak{k} \\
(N \mathbb{Z} \rightarrow 0, (A_{i_1}, \ldots, A_{i_N}) \rightarrow \text{Tr}(A_{\hat{i}_1} \cdots A_{\hat{i}_N}))
\]

\( G = \text{Sp}_{2n} \) case

The key step is the construction of the symplectic version of the Deligne map \( s: C^d_g//G \rightarrow t^d/W \).
Let \( J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} \)  
\( \Theta(\cdot) = J \cdot (\cdot)^{-1} \cdot J^{-1} \)

\( \text{Sp}_{2n} = \{ g \in G_l(2n) \mid \Theta(g) = g \} \)

\( g = \text{Sp}_{2n} = \{ g \in G_l(2n) \mid t \times J = -J \times t \} \)

\( t = \begin{bmatrix} t_1 & & & \\ & \ddots & & \\ & & -t_1 & \\ & & & -t_n \end{bmatrix} \)

\( W = S_n \times \{ \pm 1 \} \)

Cartan decomposition: \( gl_n = g \oplus g_1 \)

where \( g_1 = \{ g \in gl_n \mid t \times J = J \times t \} \)

Fact: For \( x, y \in g \) (or \( x, y \in g_1 \)), \( xy = yx \)

then \( xy = yx \in g_1 \)

Fact: \( g_1 \xrightarrow{\text{alt}_n} alt_{2n} = \{ x \in gl_n \mid t \times x = -x \} \)

\( y \rightarrow J \cdot y \)

and the Pfaffian \( \text{pf}: alt_{2n} \rightarrow \mathbb{R} \) defines a polynomial map \( \mathbf{N}: g_1 \xrightarrow{\text{alt}_n} \mathbb{R} \) of homogeneous degree \( n \).
Fact (multiplicative property of $N : g_1 \to \mathbb{R}$)
For $x, y \in g_1(\mathbb{R})$, $xy = yx$

$\implies N(xy) = N(x)N(y)$

Construction of symplectic Deligne map:
Let $X = (x_1, \ldots, x_d) \in \Sigma_{g_1}^d(\mathbb{R})$

$\implies \mathbb{R}[x_1, \ldots, x_d] \to g_{2n}(\mathbb{R})$

Let $U \subseteq \mathbb{R}[x_1, \ldots, x_d]$

$\implies \mathbb{R}[x_1, \ldots, x_d]_{\text{even}} \to \mathcal{F}_{g_1}(\mathbb{R}) \xrightarrow{N} \mathbb{R}$

$\xrightarrow{f \circ \Theta \circ \Phi}$

$\xrightarrow{f \circ \Theta \circ \Phi \circ \varphi \circ \pi}$

$\implies \text{Get a } \mathbb{R}_i\text{-point } \mathcal{F}_{g_1}(\mathbb{R}) \xrightarrow{\varphi}$

$\alpha : \text{spec } \mathbb{R} \to \text{spec } \left( \left( \mathbb{R}[x_1, \ldots, x_d] \otimes_{\mathbb{Z}} \mathbb{C} \right)^{\otimes n} \right)_{SN} \mathbb{A}(\mathbb{C})^n = \frac{d}{W}$
This finishes the construction of the symplectic Deligne map \( G^d \rightarrow t/W \) which gives rise to a map \( s: G^d / G \rightarrow t/W \).

Finally, one uses Procesi’s result for symplectic groups to show that \( s \) is the inverse of \( c: t/W \rightarrow G^d / G \).

**Remark:** Let \( G_{x_1}^2 = \{ (x,y) \in G^2 \mid [x,y] = 0 \} \).

If we know \( G_{x_1}^2 \) is reduced \( \Rightarrow \) key fact 
\[
N(xy) = N(x)N(y)
\]