Drinfeld-Gaitsgory-Vinberg interpolation Grassmannian and geometric Satake equivalence.

joint work with
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0) Affine Grassmannian
1) Geometric Satake
2) $U(\mathfrak{n}^\vee)$ via semi-infinite orbits
3) Comultiplication on $U(\mathfrak{n}^\vee)$ via Zastava
4) Schieder's deformation of zastava
5) Multiplication on $U(\mathfrak{n}^\vee)$ via Schieder's deform.
6) Schieder's conjecture and „explicit" geometric Satake.
7)* Comments/Questions
8)** More proofs?

Affine Grassmannian

G - reductive$/\mathbb{C}$ ($G^{der} = [G,G] \leftarrow$ simply connected)

$\mathcal{O} := \mathbb{C}[[z]] \leftarrow$ power series, $\mathcal{K} := \mathbb{C}((z)) \leftarrow$ Loran series

$$Gr_G := G(\mathcal{K})/G(\mathcal{O}) \simeq \left\{ (\overset{\text{G-bundle on } \mathbb{P}^1}{\mathcal{E}}, \sigma) \;\middle|\; \mathcal{E}^{triv}|_{\mathbb{P}^1\setminus\{0\}} \overset{\sigma}{\xrightarrow{\sim}} \mathcal{E}|_{\mathbb{P}^1\setminus\{0\}} \right\}$$

Thm.

can replace $\mathbb{P}^1$ by any smooth (projective) curve X

Have an action $G(\mathcal{O}) \subset G(\mathcal{K}) \curvearrowright Gr_G$ via left mult.

Notations: $\begin{matrix} T \subset & & \\ U \subset & B \subset G \end{matrix}$, Λ ← cocharacter lattice of T

max torus, unip. radical, Borel

Λ^+ ← dominant cochar.

$$H \in \Lambda = \mathrm{Hom}(\mathbb{C}^\times, T) \subset T(\mathbb{C}(z)) \subset T(\mathcal{K})$$

$$\rightsquigarrow z^H \in \mathrm{Gr}_G$$

← For $G = GL_n$, $H = (d_1, \dots, d_n)$ have $z^H = \begin{pmatrix} z^{d_1} & \cdots & 0 \\ 0 & \ddots & \vdots \\ \vdots & & \\ 0 & \cdots 0 & z^{d_n} \end{pmatrix}$

$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{t} \oplus \mathfrak{n}$

$\mathfrak{g}^\vee = \mathfrak{n}_-^\vee \oplus \mathfrak{t}^\vee \oplus \mathfrak{n}^\vee$ ← Langlands dual

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Geometric Satake + main goal

$\mathrm{Perv}_{G(\mathcal{O})} \mathrm{Gr}_G$ ← category of $G(\mathcal{O})$-equiv. perverse sheaves on Gr_G

Remark: $\mathrm{Gr}_G = \bigsqcup_{\lambda \in \Lambda^+} G(\mathcal{O}) \cdot z^\lambda$, here $\mathrm{Gr}_G^\lambda := G(\mathcal{O}) \cdot z^\lambda$

„Parabolic Bruhat decomp" (Cartan decomp.)

smooth variety

$\overline{\mathrm{Gr}_G^\lambda}$ ← projective variety

So $\mathrm{Gr}_G = \bigcup_\lambda \overline{\mathrm{Gr}_G^\lambda}$ ← ind-projective scheme

If $\mathcal{F} \in \mathrm{Perv}_{G(\mathcal{O})} \mathrm{Gr}_G$ then $\mathrm{Supp}(\mathcal{F}) \subset \overline{\mathrm{Gr}_G^{\lambda}}$

$\mathrm{Perv} = \mathrm{Perv}_{G(\mathcal{O})} \mathrm{Gr}_G \leftarrow$ Tannakian

has $\otimes$ (can convolve)

has fiber funct.

$(H^*: \mathrm{Perv} \to \mathrm{Vect})$

Thm [Lusztig+Beilinson-Drinfeld-Ginzburg-Mirković-Vilonen]

$$\begin{tikzcd}
\mathrm{Perv}_{G(\mathcal{O})} \mathrm{Gr}_G \arrow[rr, "\exists S_G", "\sim"'] \arrow[dr, "H^*"'] \arrow[rr, phantom, shift right=6ex, "\circlearrowright"] & & \mathrm{Rep}_{f.d.} G^{\vee} \arrow[dl, "\mathrm{Forg}"] \\
& \mathrm{Vect} &
\end{tikzcd}$$

S_G is called „geometric Satake isomorphism"

Langlands dual grp.

Simple example: $G = T \Rightarrow \mathrm{Gr}_G = \Lambda$, $S_G(\delta_\lambda) = \mathbb{C}_\lambda \circlearrowleft T^{\vee}$

So $\forall \mathcal{F} \in \mathrm{Perv}_{G(\mathcal{O})} \mathrm{Gr}_G \rightsquigarrow \mathfrak{g}^{\vee} \curvearrowright H^*(\mathrm{Gr}_G, \mathcal{F})$

Our goal: to construct this action.

Remark: $\mathfrak{g}^{\vee} = \mathfrak{n}_-^{\vee} \oplus \mathfrak{t}^{\vee} \oplus \mathfrak{n}^{\vee}$ then

a) $\mathfrak{t}^\vee$ - action ← from grading

b) enough construct $\mathfrak{n}^\vee$ - action ($\mathfrak{n}_-^\vee$ - using Lefschetz duality for $\mathcal{J}_{\mathrm{Gr}_G}(1)$)

Reformulation:

construct $U(\mathfrak{n}^\vee) \overset{(*)}{\curvearrowright} H^*(\mathrm{Gr}_G, \mathcal{S})$

Idea: ① Construct $U(\mathfrak{n}^\vee)$ „geometrically" as a bialgebra.

② Define $(*)$ via this construction

$U(\mathfrak{n}^\vee)$ via semi-infinite orbits

$$\mu \in \Lambda \rightsquigarrow \quad S_\mu := U(K)\cdot z^\mu \subset \mathrm{Gr}_G$$

$$T_\mu := \underset{\text{opposite to } U}{U_-}(K)\cdot z^\mu \subset \mathrm{Gr}_G$$

$$U(\mathbb{C}[z^{-1}]) \xrightarrow{ev_\infty} U$$

Remark:

$$S_\mu \xrightarrow[\sim]{z^{-\mu}\cdot} S_0 = U(K)/U(\mathcal{O}) \simeq \overset{\overset{\ker ev_\infty}{\shortparallel}}{U[z^{-1}]_1} \simeq \mathbb{A}^\infty$$

(same for T_μ)

Thm [Braverman-Finkelberg-Gaitsgory-Mirković]

$$U(\mathfrak{n}^\vee) \overset{\exists}{\simeq} \bigoplus_{\mu\in\Lambda} \underbrace{H_c^{top}(S_\mu \cap T_0, \mathbb{C})}_{=:\,\mathcal{A}}$$

of vector spaces ($T^\vee$-graded)

Langlands dual torus

$T^\vee$-graded ie $U(\mathfrak{n}^\vee)_\mu \simeq H_c^{top}(\underbrace{S_\mu \cap T_0}_{=:\,\mathcal{A}_\mu})$

Prf (rough): use geometric Satake

$+$

$U(\mathfrak{n}^\vee)_\mu \simeq V^\lambda_\mu$ for $\lambda \gg 0$

$$S_\mu \cap T_{\mu'} \simeq S_{\mu-\mu'} \cap T_0$$

Comultiplication on $\mathcal{A}$ via Zastava

$T \twoheadleftarrow B$

Have
$$\begin{tikzcd}
\mathrm{Gr}_T & \mathrm{Gr}_B \arrow[l] \arrow[r] & \mathrm{Gr}_G \\
\cup & \cup & \cup \\
Z^H & U(K)\cdot Z^H \arrow[l] \arrow[r, "\sim"] & S_H
\end{tikzcd}$$

B-bundles of degree H

iso.

S_H

G-bundle on $\mathbb{P}^1$

T_0

So $S_H \cap T_0 = \left\{ \mathcal{E}_B \subset \mathcal{E} \supset \mathcal{E}_{U_-} \right\}$

(U_-subbundle)

U_- - structure

B-structure of degree H

transversal on $\mathbb{P}^1 \setminus \{0\}$

Rem: $\mathcal{E}_B$, $\mathcal{E}_{U_-}$ are transversal on $\mathbb{P}^1 \setminus \{0\}$ $\rightsquigarrow$ trivializ. of $\mathcal{E}|_{\mathbb{P}^1 \setminus \{0\}}$

Global version:

$$\overset{\circ}{Z}{}^H = \left\{ \mathcal{E}_B \subset \mathcal{E} \supset \mathcal{E}_{U_-} \right\}$$

zastava

transversal

Reformulation: $\mathring{Z} = \mathrm{Map}_{gen}\left(\mathbb{P}^1, G/(B\times U_-) \ni pt\right)$

SL_2-example: $H \leftrightarrow n$

rk$=2$ + triv. of $\det \mathcal{E}$

$$S_n \cap T_0 = \left\{ \begin{tikzcd} \mathcal{O}(-n) \arrow[r, hook] \arrow[rr, bend right] & \mathcal{E} \arrow[r, two heads] & \mathcal{O} \end{tikzcd}, \ \det\mathcal{E} \simeq \mathcal{O}_{\mathbb{P}^1} \right\}$$

restriction to $\mathbb{P}^1 \setminus \{0\}$ is $\mathrm{id}_{\mathcal{O}_{\mathbb{P}^1\setminus\{0\}}}$

$$\mathring{Z}^n = \left\{ \begin{tikzcd} \mathcal{L} \arrow[r, hook, "\eta"] \arrow[rr, bend right] & \mathcal{E} \arrow[r, two heads, "\xi"] & \mathcal{O} \end{tikzcd}, \ \det\mathcal{E} \simeq \mathcal{O}_{\mathbb{P}^1}, \ \deg \mathcal{L} = -n \right\}$$

iso generically <=> embedding (of sheaves)

Rem: $\mathcal{L} \simeq \mathcal{O}(-n)$, $\mathcal{E} \simeq \mathcal{O} \oplus \mathcal{O}$, $\xi \simeq$ projection on second

Then $\eta \rightsquigarrow \begin{pmatrix} P \\ Q \end{pmatrix} \overset{\frown}{\in} \mathbb{C}[x,y]_n$, $Q \neq 0$, s.t. $(P,Q)=1$ i.e. $\eta \leftrightarrow \frac{P}{Q}$

Factorization:

$$\begin{tikzcd} \mathring{Z}^n \arrow[r, "\pi_n"] & \mathbb{P}^{1(n)} = \mathrm{Div}_n(\mathbb{P}^1) \end{tikzcd}$$

$$\begin{tikzcd} p \arrow[r, mapsto] & \mathrm{div}(\xi \circ \eta) \end{tikzcd}$$

$$\left(\frac{P}{Q} \longmapsto \{\text{roots of } Q\} \right)$$

Claim: $\pi_n^{-1}(n\cdot 0) = S_n \cap T_0$ (follows from the def.)

In general have

$$\begin{tikzcd}
\overset{\circ}{Z}^H \arrow[r, "\pi_H"] & \mathbb{P}^{1,(H)} = \mathrm{Div}_H(\mathbb{P}^1) \\
\mathcal{F}^H := S_H \cap T_0 \arrow[u, phantom, "\cup"] \arrow[r] & H\cdot 0 \arrow[u, phantom, "\rotatebox{90}{$\in$}"]
\end{tikzcd}$$

"T-colored" divisors on $\mathbb{P}^1$ of degree H

Proposition: $D_1 \in \mathrm{Div}_{H_1}(\mathbb{P}^1)$, $D_2 \in \mathrm{Div}_{H_2}(\mathbb{P}^2)$

$$\mathrm{Supp}\, D_1 \cap \mathrm{Supp}\, D_2 = \emptyset$$

$$\Downarrow$$

Factorization property

$$\pi^{-1}_{H_1+H_2}(D_1+D_2) \simeq \pi^{-1}_{H_1}(D_1) \times \pi^{-1}_{H_2}(D_2)$$

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partial fraction decomp.

Prf. (for SL_2): $$\frac{P}{Q_1 Q_2} \overset{\exists!}{=} \frac{P_1}{Q_1} + \frac{P_2}{Q_2}$$

$P, Q_1, Q_2 \in \mathbb{C}[t]$, $(Q_1, Q_2) = 1$

$\deg Q_1 = n_1$, $\deg Q_2 = n_2$, $\deg P < n_1 + n_2$

unique with $(P_1, Q_1) = 1$

$(P_2, Q_2) = 1$

$\deg P_1 < n_1$, $\deg P_2 < n_2$

Fix $H_1 + H_2 = H$

$$\begin{tikzcd}
\boxed{\mathring{\mathcal{F}}_H} \arrow[d] & \boxed{\mathring{\mathcal{F}}_{H_1} \times \mathring{\mathcal{F}}_{H_2}} \arrow[d] \arrow[r, leftrightarrow] & {} & \boxed{\pi_H^{-1}(0)} \arrow[d] & \boxed{\pi^{-1}_{H_1}(0) \times \pi^{-1}_{H_2}(y)} \arrow[d] \arrow[r, squiggly, "\text{Factorization}"] & {} \\
0 \neq & y & & 0 \neq & y &
\end{tikzcd}$$

$0 \in \mathbb{P}^1$, $y \in \mathbb{P}^1$

$$\begin{tikzcd}
\pi_H^{-1}(\mathbb{P}^1) \arrow[r, hook] \arrow[d, "\pi_H"] & \mathring{Z}^H \arrow[d, "\pi_H"] \\
\mathbb{P}^1 \arrow[r, hook] & \mathbb{P}^{1,(H)} \\
y \arrow[u, phantom, "\rotatebox{90}{$\in$}"] \arrow[r, mapsto] & H_1\cdot 0 + H_2\cdot y \arrow[u, phantom, "\rotatebox{90}{$\in$}"]
\end{tikzcd}$$

generic fiber $\to (S_{H_1} \cap T_0) \times (S_{H_2} \cap T_0)$

Get family over $\mathbb{P}^1$ with special fiber $\to S_{H_1+H_2} \cap T_0$

Cospecialization

$$\begin{tikzcd} S \arrow[d, "\pi"'] \\ \mathbb{P}^1 \end{tikzcd}$$

← one-parametric family

$S|_{\mathbb{P}^1\setminus\{0\}}$ ← trivial: $S|_{\mathbb{P}^1\setminus\{0\}} \simeq S|_y \times (\mathbb{P}^1\setminus\{0\})$

for some $y \in \mathbb{P}^1\setminus\{0\}$

⇝

Have cospecialization:

$$H^*_c(S|_0, \mathbb{C}) \to H^*_c(S|_y, \mathbb{C})$$

Construction: take $\pi_!\mathbb{C}_S$ ← use that it is locally constant w.r.t. $\mathbb{P}^1 = (\mathbb{P}^1\setminus\{0\}) \sqcup \{0\}$

Rem: more generally:

Q - stratified, F - constr. complex on Q

$Q = \bigsqcup_i Q_i$; F ← constant on Q_i

⇝

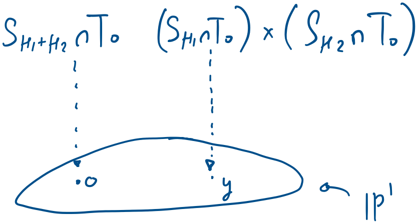
$F_{x_i} \to F_{x_j}$ if $x_i \in Q_i$, $x_j \in Q_j$

$Q_i \subset \overline{Q_j}$

Maps are compatible with inclusions:

$$\overline{Q}_i \subset \overline{Q}_j \subset \overline{Q}_k$$

Return to our situation:



Cospecialization

$$\begin{tikzcd} H_c^{top}(S_{H_1+H_2}\cap T_0,\mathbb{C}) \arrow[r] \arrow[d, equal] & H_c^{top}(S_{H_1}\cap T_0,\mathbb{C})\otimes H_c^{top}(S_{H_2}\cap T_0,\mathbb{C}) \arrow[d, equal] \\ \mathcal{A}_{H_1+H_2} \arrow[r] & \mathcal{A}_{H_1}\otimes\mathcal{A}_{H_2} \end{tikzcd}$$

$$\boxed{\mathcal{A} \xrightarrow{\Delta} \mathcal{A}\otimes\mathcal{A}}$$

comultip. on $\mathcal{A}$

Schieder's deform. of Zastava

Goal: define mult. (on $\mathcal{A}$) similarly.

Solution: Find appropriate deform. of $\mathring{Z}^H$

Recall for SL_2 have

$$\mathring{Z}^n = \{ \mathcal{L} \hookrightarrow \mathcal{E} \twoheadrightarrow \mathcal{O} \mid \text{conditions} \}$$

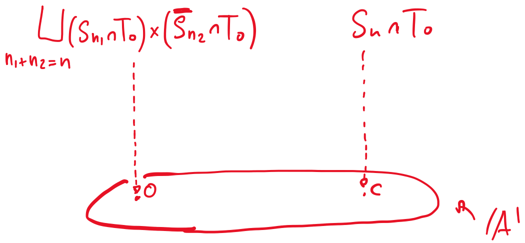
Can deform $\mathcal{E}$ itself:

$$\begin{tikzcd}
\mathring{Y}^n \arrow[d, "v_n"] & = \left\{ \mathcal{L} \overset{\eta}{\hookrightarrow} \mathcal{E}_1 \xrightarrow{\varphi} \mathcal{E}_2 \overset{\xi}{\twoheadrightarrow} \mathcal{O} \;\middle|\; \begin{array}{l} \text{trivialized} \\ \det\mathcal{E}_1, \det\mathcal{E}_2 \\ \xi\circ\varphi\circ\eta \\ \text{iso. generically} \\ \deg\mathcal{L} = -n \end{array} \right\} \arrow[d, mapsto] \\
\mathbb{A}^1 \times (\mathbb{P}^1)^{(n)} & \ni (\det\varphi, \operatorname{div}(\xi\circ\varphi\circ\eta))
\end{tikzcd}$$

Then deform. of $\mathring{\mathcal{F}}_n = S_n \cap T_0$:

$$\begin{tikzcd}
\mathring{\mathcal{Y}}^n \arrow[d, "\det\varphi"'] = \{ & \mathcal{O}(-n) \arrow[r, hook, "\eta"] \arrow[rrr, bend right, "\mathrm{id}_{\mathbb{P}^1\setminus\{0\}}"'] & \mathcal{E}_1 \arrow[r, "\varphi"] & \mathcal{E}_2 \arrow[r, two heads, "\xi"] & \mathcal{O} \mid \text{conditions} \} \\
\mathbb{A}^1
\end{tikzcd}$$

Prop [S. Schieder]


$\bigsqcup_{n_1+n_2=n} (S_{n_1}\cap T_0)\times(\overline{S}_{n_2}\cap T_0)$
$S_n \cap T_0$
0
c
$\mathbb{A}^1$

Prf. $\det\varphi = c \neq 0 \Rightarrow \varphi\text{-iso} \Rightarrow$ get $S_n \cap T_0$ ✓

$\det \varphi = 0 \Rightarrow$ can decompose φ

may be not an embedding of vector bundles

$$\begin{tikzcd}
\mathcal{O}(-n) \arrow[r, hook] & \mathcal{E}_1 \arrow[r, two heads] \arrow[rr, bend left, "\varphi"] & \mathcal{O}(-n_2) \arrow[r, hook] & \mathcal{E}_2 \arrow[r, two heads] & \mathcal{O}
\end{tikzcd}$$

data of $\overline{S}_{n_2} \cap T_0$

data of

$$\begin{tikzcd}
S_n \cap T_{n_2} \arrow[d, phantom, "\cong"', "z^{n_2}\cdot"] \\
S_{n_1} \cap T_0
\end{tikzcd}$$

Cospecialization: $\bigoplus_{n_1+n_2} H_c^{top}(\mathring{\mathcal{F}}_{n_1}) \otimes H_c^{top}(\mathcal{F}_{n_2}) \to H_c^{top}(\mathcal{F}_{n_1+n_2})$

$$\begin{tikzcd}
\bigoplus_{n_1+n_2} H_c^{top}(\mathring{\mathcal{F}}_{n_1}) \otimes H_c^{top}(\mathcal{F}_{n_2}) \arrow[r] \arrow[d, phantom, "\wr"] & H_c^{top}(\mathcal{F}_{n_1+n_2}) \\
\mathcal{A} \otimes \mathcal{A} \arrow[r, "m"] & \mathcal{A}
\end{tikzcd}$$

desired multiplication

Schieder's deformation in general: (has nice Tannakian description, do not have time)

Recall

$$\mathring{Z} = \mathrm{Maps}_{gen}(\mathbb{P}^1, G/(B \times U_-) \supset pt)$$

Can deform the group G here:

$$\begin{tikzcd}
\mathrm{Vin}_G \arrow[d, shift right] & (G \times T)/Z(G) \arrow[l, phantom, "\supset"] \arrow[d] \\
{} \arrow[u, shift right, "s"'] & {}
\end{tikzcd}$$

Vinberg semigroup

$(r = \dim T/Z(G))$

$$\begin{tikzcd} \mathbb{A}^r \supset T_{ad} & T/Z(G) \arrow[l, dotted] \end{tikzcd}$$

Construction:

$$\mathrm{Vin}_G = \mathrm{Spec}\Big(\underbrace{\bigoplus_{\lambda^\vee \in \Lambda^\vee} \mathbb{C}[G]}_{\text{Rees algebra}} \underbrace{{}_{\le \lambda^\vee}}_{\text{„Peter-Weyl" filtration}} t^{\lambda^\vee} \Big)$$

Set $\mathrm{Vin}_G^{Bruhat} := (B \times U_-) \cdot s(\mathbb{A}^r)$

$$\mathring{Y} = \mathrm{Maps}_{gen}\left(\mathbb{P}^1, \mathrm{Vin}_G/(B\times U_-) \supset \mathrm{Vin}_G^{Bruhat}/(B\times U_-)\right)$$

SL_2 - example:

$\mathbb{C}^\times \curvearrowright \mathrm{Mat}_{2\times 2}$

$t \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ta & tb \\ tc & td \end{pmatrix}$

$\rightsquigarrow$

$\mathbb{C}^\times \curvearrowright \mathbb{A}^1$

$t \cdot c = t^{②} c$

$$\begin{tikzcd} \mathrm{Vin}_{SL_2} = \mathrm{Mat}_{2\times 2} \arrow[d, shift right=1.5, "\det"'] & \supset & SL_2 \arrow[d] \\ \mathbb{A}^1 \arrow[u, hook, shift right=1.5, "\begin{smallmatrix} \left(\begin{smallmatrix} 1 & 0 \\ 0 & c \end{smallmatrix}\right) \\ \uparrow \\ c \end{smallmatrix}"'] & \ni & 1 \end{tikzcd}$$

Schieder's conjecture and „explicit" Satake

Recall the bialgebra $(\mathcal{A},\Delta,m) = \bigoplus_{H} H_c^{top}(\mathring{\mathcal{F}}^{H})$

Conjecture [Schieder]

$$(\mathcal{A},\Delta,m) \simeq U(\mathfrak{n}^\vee)$$

as bialgebras

How to prove:

Need iso. of bialgebras $\underline{U(\mathfrak{n}^\vee) \simeq \mathcal{A}}$

Schould correspond to

$\mathfrak{n}^\vee \simeq \mathfrak{a}$ primitive elements in $\mathcal{A}$

[Comes from embedding $\underline{\mathfrak{a} \hookrightarrow \mathfrak{g}^\vee}$

Can be constructed via Tannakian approach to $\mathfrak{g}^\vee$:

$$\mathfrak{g}^\vee \simeq \mathrm{End}_{\otimes}(\mathrm{Forg})$$

$\mathrm{Forg}: \mathrm{Rep}_{f.d.} G^\vee \to \mathrm{Vect}$

So $\mathfrak{a} \to \mathfrak{g}^\vee \iff \left\{ \mathfrak{a} \curvearrowright V \quad \forall V \in \mathrm{Rep}_{f.d.}(G^\vee) \right\}$

compatible with $\otimes$

$\Updownarrow$ Satake

$$\left\{ \mathfrak{a} \curvearrowright H^*(\mathrm{Gr}_G, \mathcal{S}), \quad \forall \mathcal{S} \in \mathrm{Perv}_{G(\mathcal{O})}(\mathrm{Gr}) \right\}$$

compatible with $*$ (convolution)

natural to replace by $\mathcal{A}$

How to construct $\mathcal{A} \curvearrowright H^*(\mathrm{Gr}_G, \mathcal{S})$?

As always need to deform Gr_G.

Take $G = SL_2$:

rk $\mathcal{E} = 2$

$$\mathrm{Gr}_{SL_2} = \left\{ \begin{tikzcd} \mathcal{O} \arrow[r, dashed, "\eta"] \arrow[rr, bend right, "|_{\mathbb{P}^1 \setminus \{0\}} = \mathrm{id}"'] & \mathcal{E} \arrow[r, dashed, "\xi"] & \mathcal{O} \end{tikzcd} \;\middle|\; \begin{array}{l} \det \mathcal{E} \simeq \mathcal{O}_{\mathbb{P}^1} \\ \xi \circ \eta|_{\mathbb{P}^1 \setminus \{0\}} = \mathrm{id} \end{array} \right\}$$

rational at 0

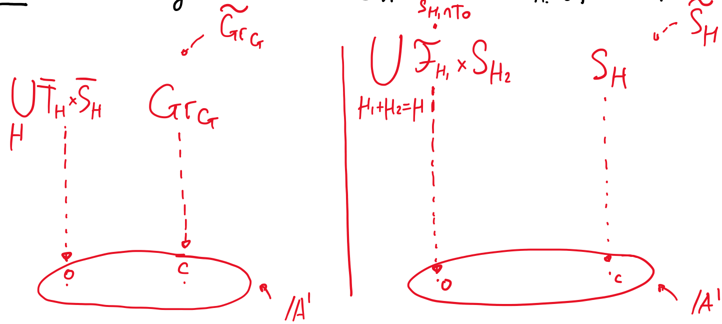
(i.e. $\eta \rightsquigarrow \mathcal{O}(-n\cdot 0) \to \mathcal{E}$
$\xi \rightsquigarrow \mathcal{E} \to \mathcal{O}(m\cdot 0)$)

trivialization via $\operatorname{im}\eta|_z \oplus \ker\xi|_z$

Can deform:

$$\begin{tikzcd} \widetilde{Gr}_G \arrow[d, "\det\varphi"'] & = \Big\{ \mathcal{O} \arrow[r, dashed, "\eta"] \arrow[rrr, bend right, "|_{\mathbb{P}^1\setminus\{0\}} = \underline{id}."'] & \mathcal{E}_1 \arrow[r, "\varphi"] & \mathcal{E}_2 \arrow[r, dashed, "\xi"] & \mathcal{O} \;\Big|\; \text{conditions} \Big\} \\ \mathbb{A}^1 \end{tikzcd}$$

Similarly define $\tilde{S}_n \leftarrow$ deform. of S_n



$\widetilde{Gr}_G$
$\bigcup_H \overline{T}_H \times \overline{S}_H$
Gr_G
0
c
$\mathbb{A}^1$
$S_{H_1} \cap T_0$
$\bigcup_{H_1+H_2=H} \mathcal{F}_{H_1} \times S_{H_2}$
S_H
$\tilde{S}_H$
0
c
$\mathbb{A}^1$

$$\mathrm{Recall:}\ \overset{\overset{V}{\parallel}}{H^*(\mathcal{S}, Gr_G)} = \bigoplus_{H\in\Lambda} \overset{\overset{V_H}{\parallel}}{H_c^*(\mathcal{S}, S_H)}$$

So want some sheaf $\tilde{\mathcal{S}}$ on $\tilde{S}_H$

$$\text{s.t.} \begin{cases} \tilde{\mathcal{S}}|_c = \mathcal{S}|_{S_H} \rightsquigarrow V_H \\ \tilde{\mathcal{S}}|_0 = \mathbb{C} \boxtimes \mathcal{S}|_{S_{H_2}} \rightsquigarrow \mathcal{A}_{H_1} \otimes V_{H_2} \end{cases}$$

Cospecialization gives action

$$\begin{tikzcd} \tilde{S}_H \arrow[r, hook, "2"] & Gr_G \times Gr_G \times \mathbb{A}^1 \end{tikzcd}$$

closed

$$\rotatebox{90}{$\in$} \qquad\qquad \rotatebox{90}{$\in$}$$

$$\{\mathcal{O} \overset{\eta}{\dashrightarrow} \mathcal{E}_1 \xrightarrow{\varphi} \mathcal{E}_2 \overset{\xi}{\dashrightarrow} \mathcal{O}\} \mapsto \{\mathcal{O} \overset{\eta}{\dashrightarrow} \mathcal{E}_1 \overset{\xi\circ\varphi}{\dashrightarrow} \mathcal{O},\ \mathcal{O} \overset{\varphi\circ\eta}{\dashrightarrow} \mathcal{E}_2 \overset{\xi}{\dashrightarrow} \mathcal{O},\ \det\varphi\}$$

Then $\tilde{\mathcal{S}} := 2^*(\mathbb{C} \boxtimes \mathcal{S} \boxtimes \mathbb{C})$

Cospecialization for $(\tilde{S}_H, \tilde{\mathcal{S}})$

$$\rightsquigarrow$$

$$\bigoplus_{H_1+H_2=H} \mathcal{A}_{H_1} \otimes V_{H_2} \to V_H$$

$$\wr$$

$$\underline{\mathcal{A} \otimes V \to V}$$

action of $\mathcal{A}$ on $H^*(Gr_G, \mathcal{S})$

$$\begin{array}{c} U(\mathfrak{n}^\vee) = \bigoplus H^0(S_H \cap T_0) \\ \shortparallel \\ \boxed{\bigoplus H^0(S_{H+0} \cap T_0)} \end{array}$$

Thm [Finkelberg-Mirković-K]

a) The bialgebra $\mathcal{A}$ is isomorphic to $U(\mathfrak{n}^\vee)$

b) The natural action $\mathcal{U}(\mathfrak{n}^{\vee}) \curvearrowright V = H^{*}(\mathrm{Gr}_{G}, \mathcal{S})$ is geometrically given by the cospecialization construction above

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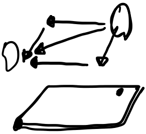
Thank you!

Ideas: All comp. follow from the fact that we can consider multi-fibrations



Comments/Questions

① Note that $\underbrace{\mathcal{U}(\mathfrak{n}^{\vee})}_{} \simeq \mathcal{A} = \bigoplus H_c^{top}(\underline{S_{\mu} \cap T_0})$

has a basis of irred. components

Also geometric Satake gives us

$$\underline{V}^{\lambda} = \bigoplus_{H} H^{top}(\mathrm{Gr}_G^{\lambda} \cap S_H, \mathbb{C})$$

has basis of irreducible components.

From Thm easy to see that basis in

V^λ is the image of basis in $U(\mathfrak{n}^\vee)$ via

$$\begin{tikzcd}
\underline{U(\mathfrak{n}^\vee)} \arrow[r, two heads] & V^\lambda \\
x \arrow[u, phantom, sloped, "\in"] \arrow[r, mapsto] & x \cdot v_- \arrow[u, phantom, sloped, "\in"]
\end{tikzcd}$$

lowest vector

(some basis elements go to 0)

So bases are compatible.

This is the case with canonical and semi-canonical bases defined by Lusztig.

Turns out that our basis is neither canonical nor semi-canonical!

(Dranowski, Kamnitzer, Morton-Ferguson)

SL_6 - counterexample

For $G = SL_2$ bases coincide:

$$U(\mathfrak{n}^\vee) = \bigoplus_{n \geq 0} \mathbb{C} \cdot e_n, \quad e_n e_m = \binom{n+m}{n} e_{n+m}$$

② Deformations $\mathring{Y}^n$, $\widetilde{Gr}_G$, $\widetilde{S}_H$...

(their open parts that are enough for our purposes)

Can be constructed via very general construction due to Drinfeld and Gaitsgory:

$$\begin{tikzcd} \mathbb{C}^\times \curvearrowright Z \arrow[r, rightsquigarrow] & \tilde{Z} \arrow[d] \\ & \mathbb{A}^1 \end{tikzcd}$$

└

$$\begin{tikzcd} \mathbb{C}^\times \curvearrowright \mathbf{X} := \mathbb{A}^2 \arrow[d] & \ni (c_1, c_2) \arrow[d, mapsto] \\ \mathbb{A}^1 & \ni c_1 c_2 \end{tikzcd}$$

$t \cdot (c_1, c_2) = (tc_1, t^{-1}c_2)$

$$\begin{tikzcd} S \arrow[d] \arrow[r, rightsquigarrow] & \mathbf{X}_S := S \times_{\mathbb{A}^1} \mathbf{X} \\ \mathbb{A}^1 \end{tikzcd}$$

Define

$$\begin{tikzcd} \tilde{Z} \arrow[d] \\ \mathbb{A}^1 \end{tikzcd}$$

as

$$\mathrm{Mor}_{\mathbb{A}^1}(S, \tilde{Z}) = \mathrm{Mor}(\mathbf{X}_S, Z)^{\mathbb{C}^\times}$$

└

Then $\tilde{Z}|_{\mathbb{G}_m} \simeq Z \times \mathbb{G}_m$

$$\tilde{Z}|_0 \simeq Z^{attr} \times_{Z^{\mathbb{C}^\times}} Z^{rep}$$

└

③ Existence of general approach to the construction of deformation allows us to define these deformations in the setting of arbitrary Kac-Moody $\mathfrak{g}$ (but we can not prove anything in this setting)

Even for affine type A where Nakajima constructed sort of geometric Satake

④ Would be nice to generalize Drinfeld-Gaitsgory to $T \curvearrowright Z$

(not nes. $T = \mathbb{C}^\times$)

$\widetilde{\mathrm{Gr}}_G$ is a deformation over $\mathbb{A}^r \supset T_{ad}$

can get the whole $\widetilde{\mathrm{Gr}}_G$ via DG?

⑤ Construction of $U_q(\mathfrak{n}^\vee)$?

Thank you!