What's special about special?

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Outline

What this talk is about

$G(\mathbb{R})$ real reductive algebraic group.

Example: $Sp(2n,\mathbb{R})$

$(\pi,\mathcal{H}_\pi)$ irreducible (usually ∞-diml) rep of $G(\mathbb{R})$.

Example: $\mathcal{H}_\pi =$ half-densities on $\mathbb{RP}^{2n-1}$.

Study $\pi \rightsquigarrow \mathrm{WF}(\pi)$, a simple geometric invariant of π.

$\mathrm{WF}(\pi) \subset \mathfrak{g}(\mathbb{R})^*$, closed $G(\mathbb{R})$-invt cone.

Ex: WF(half-dens on $\mathbb{RP}^{2n-1}$) $=$ rk ≤ 2 nilp sympl.

Ex: WF(generic irr of $Sp(2n,\mathbb{R})$) $=$ all nilp sympl.

WF encodes interesting information about π.

Easy algebra: $G(\mathbb{R})$ has finite # nilp orbits on $\mathfrak{g}(\mathbb{R})^*$.

Easy soft analysis: $\mathrm{WF}(\pi) =$ finite union of nilp orbits.

Deep result from Lusztig: π "integral" $\implies$ $\mathrm{WF}(\pi)$ special.

PLAN: sketch defs, sketch Geck, Dong-Yang integral def of special, ask for direct proof of Lusztig $\implies$ above.

Howe's wavefront set

π nice irr rep of $G(\mathbb{R})$ on Hilbert space $\mathcal{H}_\pi$.

Read Howe's beautiful Wave front sets of reps of Lie groups for def of $\mathrm{WF}(\pi) \subset \mathfrak{g}(\mathbb{R})^*$: soft analysis.

Outline: trace class ops T on $\mathcal{H}_\pi \rightsquigarrow$ "matrix coeff" distributions π_T on $G \rightsquigarrow \mathrm{WF}(\pi_T) \subset T^*G$.

Big idea for controlling $\mathrm{WF}(\pi)$:

$$z \in \text{Cent}\, U(\mathfrak{g}) \rightsquigarrow \pi(z) = \text{scalar}$$
$$\rightsquigarrow \text{differential equation } (z - \pi(z)) \cdot \pi_T = 0$$
$$\rightsquigarrow \mathrm{WF}(\pi_T) \subset \text{ zeros of symbol of } z.$$

Symbols of $z \in \text{Cent}\, U(\mathfrak{g})$ are homog polys $p \in S(\mathfrak{g})^G$.

Real nilpotent cone (where $\mathrm{WF}(\pi)$ must live!) is

$$\mathcal{N}^*_{\mathbb{R}} = \{\lambda \in \mathfrak{g}(\mathbb{R})^* \mid p(\lambda) = 0 \ (p \in S(\mathfrak{g})^G \text{ homogeneous})\}.$$

$\mathcal{N}^*_{\mathbb{R}}/G(\mathbb{R})$ finite $\implies$ $\mathrm{WF}(\pi) =$ finite # $G(\mathbb{R})$ orbs.

Calculating nilpotent orbits

WF(π) is elementary, uncomplicated invariant of π.

Zeroth problem: describe $G(\mathbb{R})$ orbits $\mathcal{O}_{\mathbb{R}}$ on $\mathfrak{g}(\mathbb{R})^*$.

$T_s(\mathbb{R}) \subset G(\mathbb{R})$ Iwasawa (most split) max torus.

$$X_{*,h}(T_s) = \mathrm{Hom}_{\mathrm{alg}}(\mathbb{R}^\times, T_s(\mathbb{R})) \subset \mathrm{Hom}_{\mathrm{alg}}(\mathbb{C}^\times, T_s) = X_*(T_s).$$

$d \in X_{*,h}(T_s) \rightsquigarrow$ Lie algebra $\mathbb{Z}$-grading

$$\mathfrak{g}(\mathbb{R}) = \sum_{n\in\mathbb{Z}} \mathfrak{g}(\mathbb{R})_d(n), \qquad \mathfrak{t}_s(\mathbb{R}) \subset \mathfrak{g}(\mathbb{R})_d(0).$$

Levi $G(\mathbb{R})^d$ has open orbits on each $\mathfrak{g}(\mathbb{R})^*_d(n)$ $(n \neq 0)$

Thm (Jacobson-Morozov) $\mathcal{O}_{\mathbb{R}} \subset \mathcal{N}^*_{\mathbb{R}} \rightsquigarrow d \in X_{*,h}(T_s)$ so

$\mathcal{O}_{\mathbb{R}}$ meets $\mathfrak{g}(\mathbb{R})^*_d(2)$ in open, $\quad d \in [\mathfrak{g}(\mathbb{R})_d(2), \mathfrak{g}(\mathbb{R})_d(-2)]$.

This defines a finite-to-one map

$$\mathcal{N}^*_{\mathbb{R}}/G(\mathbb{R}) \rightsquigarrow X_{*,h}(T_s)/W_s(\mathbb{R}) \simeq \text{dom cowts } X^+_{*,h}(T_s).$$

Fiber over $d \leftrightsquigarrow$ open orbits of $G(\mathbb{R})^d$ on $\mathfrak{g}(\mathbb{R})^*_d(2)$

Dominant coweight d called the Dynkin diagram of $\mathcal{O}_{\mathbb{R}}$.

Structure of orbits

Nilp orb $\mathcal{O}_{\mathbb{R}} \rightsquigarrow$ dom $d \in \mathrm{Hom}_{\mathrm{alg}}(\mathbb{R}^\times, T_s(\mathbb{R}))$,

$\mathcal{O}_{\mathbb{R}}$ meets $\mathfrak{g}(\mathbb{R})^*_d(2)$ in open, $\quad d \in [\mathfrak{g}(\mathbb{R})_d(2), \mathfrak{g}(\mathbb{R})_d(-2)]$.

$$\mathfrak{g}(\mathbb{R})_d(n) = \{X \in \mathfrak{g}(\mathbb{R}) \mid [d, X] = nX\}$$
$$\mathfrak{g}(\mathbb{R})^*_d(n) = [\mathfrak{g}(\mathbb{R})_d(-n)]^*.$$

If $\alpha \in R^+$ simple, then $\alpha(d) = 0$ or 1 or 2.

Partition simple roots Π as $\Pi_d(0) \cup \Pi_d(1) \cup \Pi_d(2)$.

$$G^d = \text{Levi subgp} \quad \leftrightsquigarrow \Pi_d(0)$$
$$\mathfrak{g}_d(-1) = \text{sum of } G^d \text{ irrs, hwts} \quad -\alpha \in \Pi_d(1)$$
$$P = G^d U, \qquad \mathfrak{u} = \textstyle\sum_{n>0} \mathfrak{g}_d(n).$$

Fix $\lambda \in \mathcal{O}_{\mathbb{R}} \cap \mathfrak{g}(\mathbb{R})^*_d(2)$. Then $G^\lambda \subset P$, and

$$G^\lambda = [G^d]^\lambda \cdot U^\lambda \qquad \text{(Levi decomp)}$$
$$\mathfrak{u}^\lambda = \textstyle\sum_{n>0} \mathfrak{g}^\lambda_d(n),$$

all decompositions defined over $\mathbb{R}$.

Symplectic structure on orbits

Nilp orb $\mathcal{O}_{\mathbb{R}} \rightsquigarrow$ dom $d \in \mathrm{Hom}_{\mathrm{alg}}(\mathbb{R}^\times, T_s(\mathbb{R}))$,

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$$\mathfrak{u}^\lambda = \textstyle\sum_{n>0} \mathfrak{g}^\lambda_d(n)$$

$$T_{eG^\lambda}(G \cdot \lambda) = \mathfrak{g}/\mathfrak{g}^\lambda$$

$$= \mathfrak{g}_d(-1) + \textstyle\sum_{m \geq 0} \left[\mathfrak{g}_d(-m-2) + \mathfrak{g}_d(m)/\mathfrak{g}_d(m)^\lambda\right].$$

$\mathcal{O}_{\mathbb{R}}$ is a symplectic manifold: nondegenerate form

$$\omega_\lambda \colon \mathfrak{g}(\mathbb{R})/\mathfrak{g}(\mathbb{R})^\lambda \times \mathfrak{g}(\mathbb{R})/\mathfrak{g}(\mathbb{R})^\lambda \to \mathbb{R}$$

$$\omega_\lambda(X, Y) = \lambda([X, Y])$$

$$[\mathfrak{g}_d(-m-2)]^* \simeq_{\omega_\lambda} \mathfrak{g}_d(m)/\mathfrak{g}_d(m)^\lambda \quad (m \geq 0)$$

ω_λ nondegenerate on $\mathfrak{g}_d(-1)$.

ω_λ needed to relate $\mathcal{O}_{\mathbb{R}}$ to representation theory.

Geck conj: $\mathcal{O}_{\mathbb{R}}$ special $\iff$ ω_λ integral (to be explained).

Integral structures on $\mathfrak{g}$

Integral structure on N-diml Lie algebra $\mathfrak{g}$ over char 0 field k is free rank N lattice $\mathfrak{g}_{\mathbb{Z}} \subset \mathfrak{g}$ subject to

$$\mathfrak{g} = \mathfrak{g}_{\mathbb{Z}} \otimes_{\mathbb{Z}} k, \qquad [\mathfrak{g}_{\mathbb{Z}}, \mathfrak{g}_{\mathbb{Z}}] \subset \mathfrak{g}_{\mathbb{Z}}.$$

Equivalent: basis $\{X_1, \ldots, X_N\}$ subject to

$$[X_i, X_j] = \sum_k c_{ij}^k X_k, \qquad c_{ij}^k \in \mathbb{Z}.$$

Example: $\mathfrak{g} = \mathfrak{sl}(2)$, basis (this one we'll generalize)

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

$$[H, X] = 2X, \qquad [H, Y] = -2Y, \qquad [X, Y] = H.$$

Example: $\mathfrak{g} = \mathfrak{so}(3)$, basis (but this is worth more study!)

$$U = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; V = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \; W = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix},$$

$$[U, V] = W, \qquad [V, W] = U, \qquad [W, U] = V.$$

Chevalley integral structure

$\mathfrak{g} \supset \mathfrak{b} \supset \mathfrak{t}$ cplx reduc; roots $\Delta(\mathfrak{g},\mathfrak{t}) \subset \mathfrak{t}^*$, coroots $\Delta^\vee(\mathfrak{g},\mathfrak{t}) \subset \mathfrak{t}$.

Integral structure is called split if

1. Have integral basis = basis $\{X_1,\ldots,X_\ell\}$ of $\mathfrak{t}$, root vectors X_α for each root; and
2. $[X_\alpha, X_{-\alpha}]$ is equal to the coroot $H_\alpha = \alpha^\vee$.

Chevalley: in a split integral structure, set of root vecs up to sign $\{\pm X_\alpha\}$ is determined up to $\mathrm{Ad}(T)$, so should be thought of as unique.

Still in a split integral structure,

$$\mathbb{Z}\Delta^\vee \subset \mathfrak{t}_{\mathbb{Z}} \subset \{t \in \mathfrak{t} \mid \alpha(t) \in \mathbb{Z} \quad (\alpha \in \Delta)\};$$

and any such lattice $\mathfrak{t}_{\mathbb{Z}}$ is allowed.

These $\mathfrak{t}_{\mathbb{Z}}$ are the $X_*(T) \leftrightsquigarrow$ root data for alg G, $\mathrm{Lie}(G) = \mathfrak{g}$.

If $\mathfrak{g}$ semisimple, split integral structure (unique up to $\mathrm{Ad}(T)$) with $\mathfrak{t}_{\mathbb{Z}} = \mathbb{Z}\Delta^\vee$ is the Chevalley integral structure.

Integral linear functionals

split int str $\mathfrak{g}_{\mathbb{Z}} \subset \mathfrak{g} \rightsquigarrow \mathfrak{g}^*_{\mathbb{Z}} =_{\text{def}} \mathrm{Hom}_{\mathbb{Z}}(\mathfrak{g}_{\mathbb{Z}}, \mathbb{Z}) \subset \mathfrak{g}^*$.

$\mathcal{O}_{\mathbb{R}}$ weakly integral if $\mathcal{O}_{\mathbb{R}} \cap \mathfrak{g}^*_{\mathbb{Z}} \neq \emptyset$; includes nilpotent.

Precisely: elt $d \in$ coroot lattice $\subset \mathfrak{t}_{\mathbb{Z}}$; $\mathcal{O}_{\mathbb{R}}$ has element

$$\lambda \in \mathfrak{g}^*_{d,\mathbb{Z}}(2): \quad \lambda(X_\alpha) = c_\alpha \in \mathbb{Z} \quad (\alpha \in \Delta,\ \alpha(d) = 2).$$

Symplectic form ω_λ defines

$$\omega_{\lambda,\mathbb{Z}} \colon \mathfrak{g}_{\mathbb{Z}}/\mathfrak{g}^\lambda_{\mathbb{Z}} \hookrightarrow \left[\mathfrak{g}_{\mathbb{Z}}/\mathfrak{g}^\lambda_{\mathbb{Z}}\right]^*.$$

Nondegen/$\mathbb{R}$ implies $\mathrm{im}(\omega_{\lambda,\mathbb{Z}})$ has finite index N_λ.

Weights of d decomposition factors $\omega_{\lambda,\mathbb{Z}}$ as sum of

$$\omega_{\lambda,\mathbb{Z}}(m) \colon \mathfrak{g}_{d,\mathbb{Z}}(m)/\mathfrak{g}_{d,\mathbb{Z}}(m)^\lambda \hookrightarrow [\mathfrak{g}_{d,\mathbb{Z}}(-m-2)]^* \qquad (m \geq 0),$$

$$\omega_{\lambda,\mathbb{Z}}(-1) \colon \mathfrak{g}_{d,\mathbb{Z}}(-1) \hookrightarrow \mathfrak{g}_{d,\mathbb{Z}}(-1)^*.$$

Each of these has finite index $N_\lambda(m)$ in its image, and

$$N_\lambda = N_\lambda(-1) \cdot \textstyle\prod_{m \geq 0} N_\lambda(m).$$

λ is strongly integral if $N_\lambda = 1$; that is, if $\omega_{\lambda,\mathbb{Z}}$ is nondeg/$\mathbb{Z}$.

λ is Geck integral if $N_\lambda(-1) = 1$; that is, if $\omega_{\lambda,\mathbb{Z}}(-1)$ is nondeg/$\mathbb{Z}$.

Lusztig's notion of special in $\widehat{W}$

W Weyl grp for $G \rightsquigarrow$ Chevalley group $G(\mathbb{F}_q) \supset B(\mathbb{F}_q)$.

Natural bij $\sigma \leftrightsquigarrow \pi_q(\sigma)$ between irrs $\sigma \in \widehat{W}$ and irrs $\pi_q(\sigma)$ of $G(\mathbb{F}_q)$ appearing in fns on $G(\mathbb{F}_q)/B(\mathbb{F}_q)$.

generic degree $\widetilde{P}_\sigma(q) =_{\text{def}} \dim \pi_q(\sigma)$: poly in q, $\mathbb{Q}$-coeffs.

Cpt mfld $X = G(\mathbb{C})/B(\mathbb{C})$: cohom only even degs.

W acts naturally on $H^*(X)$. $\rightsquigarrow$ regular rep of W.

Can therefore define fake degree

$$P_\sigma(q) = \sum_{i=0}^r (\text{mult of } \sigma \text{ in } H^{2i}(X))q^i$$

poly in q, nonneg integer coeffs summing to $\dim \sigma$.

$G = GL(n), \quad W = S_n: \quad \widetilde{P}_\sigma = P_\sigma$.

Define $\widetilde{a}_\sigma$ = least q^a in $\widetilde{P}_\sigma$, a_σ = least q^a in P_σ.

Lusztig 1979: $\widetilde{a}_\sigma \leq a_\sigma$; say σ is special if $\widetilde{a}_\sigma = a_\sigma$.

Lusztig's notion of special for nilpotent orbits

Springer (1978) defined inclusion j

$$j\colon \text{nilpotent orbits in } \mathfrak{g}^* \hookrightarrow \widehat{W}, \qquad \mathcal{O} \mapsto j(\mathcal{O}).$$

Easy: $\dim(\mathcal{O}) = 2r - 2a_{j(\mathcal{O})}$ ($r = \#\text{pos roots}$).

Springer (1978) also defined surjection p ($p \circ j = id$)

$$p\colon \widehat{W} \twoheadrightarrow \text{nilpotent orbits in } \mathfrak{g}^*, \qquad \sigma \mapsto p(\sigma).$$

Easy: $\dim(p(\sigma)) \geq 2r - 2a_\sigma$, equality iff $j \circ p(\sigma)) = \sigma$.

KL theory partitions $\widehat{W}$ in families ($\leftrightsquigarrow$ two-sided cells).

Theorem (Lusztig)

1. Each family $\mathcal{F} \subset \widehat{W}$ has unique special rep $\sigma_s(\mathcal{F})$.
2. Function $\widetilde{a}_\sigma$ is constant on each family.
3. Function a_σ has unique minimum on $\mathcal{F}$, at $\sigma_s(\mathcal{F})$.
4. $\sigma_s(\mathcal{F})$ is $j(\mathcal{O}(\mathcal{F}))$, special nilpotent orbit.

Geck conj/Dong-Yang thm on special nilps

$G \supset B \supset T$, $\mathcal{O} \subset \mathfrak{g}^*$ $\rightsquigarrow$ Jacobson-Morozov dom $d \in X_*(T)$:

$$d \in [\mathfrak{g}_d(2), \mathfrak{g}_d(-2)], \quad \mathcal{O} \cap \mathfrak{g}_d^*(2) \text{ open in } \mathfrak{g}_d^*(2)$$

$\rightsquigarrow \omega_\lambda$ symplectic on $\mathfrak{g}/\mathfrak{g}^\lambda$, $\omega_\lambda(-1)$ on $\mathfrak{g}_d(-1) \subset \mathfrak{g}/\mathfrak{g}^\lambda$.

Fix also split int str $\mathfrak{g}_{\mathbb{Z}} \subset \mathfrak{g} \rightsquigarrow \mathfrak{g}_{\mathbb{Z}}^* \subset \mathfrak{g}^*$.

May choose representative $\lambda_{\mathbb{Z}} \in \mathcal{O} \cap \mathfrak{g}_{d,\mathbb{Z}}^*(2)$.

Conj (Geck 2018) $\mathcal{O}$ special iff $\exists \lambda_{\mathbb{Z}}$ so $\omega_{\lambda_{\mathbb{Z}}}(-1)$ nondeg/$\mathbb{Z}$.

Proved by Geck (types *EFG*), Dong-Yang (2019) (types *ABCD*).

Proof is case-by-case using enumeration of special nilps.

Recall that hypothesis Geck integral in Geck conjecture is weaker than natural hypothesis strongly integral.

Hope: Geck integral equivalent to strongly integral.

Lusztig thm on special nilps

Theorem (Lusztig) Suppose π irr rep of real reductive $G(\mathbb{R})$ of integral infl char. Then there is a special $\mathcal{O} \subset \mathfrak{g}^*$ so that $\mathrm{WF}(\pi)$ is closure of some real forms $\mathcal{O}^j_{\mathbb{R}}$ of $\mathcal{O}$.

Proof is by KL theory, properties of families in $\widehat{W}$.

Hope (point of talk): there is a conceptual path

$$\pi \text{ integral infl char} \rightsquigarrow \mathrm{WF}(\pi) \text{ strongly integral.}$$

Such a path could give a conceptual proof

$$(\mathcal{O} \text{ special}) \implies (\mathcal{O} \text{ str int}) \implies (\mathcal{O} \text{ Geck int}).$$

which is half of Geck's conjecture.

Path to Hope: $\exists$? nice $\mathbb{Z}$-forms of reps with int infl char.

I like this question. Can find in *Green Monster* (Vogan 1981) $\mathbb{Z}$-forms for $SL(2,\mathbb{R})$ reps in block of finite-diml.

First easy exercise: other blocks of int infl char. for $SL(2,\mathbb{R})$?

Thank you!