Atlas of Lie Groups and Representations

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Atlas of Lie Groups and Representations
Unipotent Representations

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The Unitary Dual

G : real reductive group $(\mathrm{GL}(n,\mathbb{R}), \mathrm{Sp}(2n,\mathbb{R}), \mathrm{E}_8(\mathit{split}))$

Problem: compute the Unitary Dual $\widehat{G}$ of G: the irreducible, norm-preserving representations of G on a Hilbert space (up to isomorphism)

Hermann Weyl (1920s)

Not known in general. Some cases which are known:

Compact groups (Weyl)

$\mathrm{SL}(2,\mathbb{R})$ (Bargmann 1947)

$\mathrm{GL}(n,\mathbb{R}), \mathrm{GL}(n,\mathbb{C}), \mathrm{GL}(n,\mathbb{H})$ (Vogan, 1986)

Complex classical groups (Barbasch 1989)

G_2 (Vogan, 1994)

The answer is known to be very complicated in many cases.

Atlas of Lie Groups and Representations

Theorem (Vogan, 1980s)

For any given group G there exists a finite algorithm to compute $\widehat{G}$

Question: is this result of more than theoretical interest? In other words, can this algorithm be made explicit, and implemented on a computer?

Atlas of Lie Groups and Representations (2002): take this question seriously...

Also:

- many other questions in representation theory.
- Software for the mathematical community
- I only really understand something if I can implement it on a computer

$(\mathfrak{g}, K)$-modules

Harish-Chandra reformulated the question algebraically in terms of $(\mathfrak{g}, K)$-modules.

$\mathfrak{g}_0 = \mathrm{Lie}(G), \mathfrak{g} = \mathfrak{g}_0 \otimes \mathbb{C}$, K : maximal compact subgroup of G.

An admissible $(\mathfrak{g}, K)$-module is an algebraic representation (π, V) of $\mathfrak{g}$ and (compatibly) K. The admissible dual is known (Langlands/Knapp/Zuckerman/Vogan).

We say (π, V) (irreducible) is Hermitian if it preserves a non-degenerate invariant Hermitian form $\langle\, , \,\rangle$ on V:

$$\begin{aligned} \langle \pi(X)v, w\rangle + \langle x, \pi(X)w\rangle \quad (X \in \mathfrak{g}_0) \\ \langle \pi(k)v, \pi(k)w\rangle = \langle v, w\rangle \quad (k \in K) \end{aligned}$$

We say (π, V) is unitary if this form is positive definite.

Atlas Algorithm

Theorem (Harish-Chandra)

$$\widehat{G} \xleftrightarrow{1-1} \{(\pi, V) \quad \textit{irreducible unitary}\}/\sim \textit{equivalence}$$

The groups: $G(\mathbb{C})$ is a connected, complex reductive group; with real points $G = G(\mathbb{R})$

Theorem (Atlas)

Suppose π is an irreducible admissible $(\mathfrak{g}, K)$-module. There is an explicit, computable algorithm to compute the signature of the invariant Hermitian form on V, and in particular to determine if π is unitary.

This algorithm has been implemented in the `ATLAS` software.

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UNITARY REPRESENTATIONS OF REAL REDUCTIVE GROUPS

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Signatures of Hermitian forms

The notion that an invariant Hermitian form on an infinite dimensional vector space is positive definite is well defined.

However, what does it mean to compute the signature of a Hermitian form $\langle\, , \rangle$ on (π, V)?

$$\pi|_K \simeq \sum_{\mu \in \widehat{K}} \mathrm{mult}_\pi(\mu)\mu \quad (\mathrm{mult}_\pi(\mu) \in \mathbb{Z}_{\geq 0})$$

This sum is orthogonal with respect to $\langle\, , \rangle$. On the μ K-isotypic component:

$$\langle\, , \rangle \text{ is } \begin{cases} \text{positive definite on} & p \text{ copies of } \mu \\ \text{negative definite on} & q \text{ copies of } \mu \end{cases}$$

$(p + q = \mathrm{mult}_\pi(\mu))$.

The ring $\mathbb{W}$

Record this information:

Definition

$\mathbb{W} = \mathbb{Z}[s] \quad (s^2 = 1)$

$\text{sig}_\pi : \widehat{K} \to \mathbb{W} : \text{sig}_\pi(\mu) = p + qs$

$$(\pi, \langle\, , \rangle) = \sum_{\mu \in \widehat{K}} \text{sig}_\pi(\mu)\mu$$

sig_π is a refinement of the mult_π:

$$\text{sig}_\pi(\mu)|_{s=1} = p + q = \text{mult}_\pi(\mu) \quad (\mu \in \widehat{K})$$

Theorem (Vogan)

Suppose π is an irreducible admissible representation. Then there are finitely many irreducible tempered representations π_i, and $w_i \in \mathbb{W}$ such that

$$sig_\pi = \sum_{i=1}^{n} w_i mult_{\pi_i}$$

π is unitary if and only if $w_i \in \mathbb{Z}$ for all i, or $w_i \in s\mathbb{Z}$ for all i.

In other words the sig_π functions are in the finite $\mathbb{W}$-span of the mult_{π_i} functions. Or:

$$\mathrm{sig}_\pi(\mu) = \sum_{i=1}^{n} \mathrm{mult}_{\pi_i}(\mu) w_i$$

Tempered implies unitary.

The functions mult_{π_i} are "known".

`ATLAS` computes mult_π for any π (tempered or not; this is a long story in itself)

Existence and uniqueness of Hermitian forms

Question: suppose (π, V) is an irreducible admissible representation.

- Does it have an invariant Hermitian form?
- If so, it is canonical up to sign?

Answers: No, and No   (i.e. not always)

From now on: restrict to case of real infinitesimal character ($\lambda \in X^* \otimes \mathbb{R}$). There is a natural reduction to this case (Vogan$\leftrightarrow$Knapp).

Knapp and Zuckerman (1986) computed the "Hermitian" dual.

Theorem (Atlas)

Let θ be the Cartan involution of $\mathfrak{g}$. Then π is Hermitian if and only if $\pi^\theta \simeq \pi$.
In particular: every representation of an equal rank group is Hermitian.

(Reminder: real infinitesimal character)

The c-form

Example: $G = SL(2,\mathbb{R})$, principal series with odd K-types $\leftrightarrow 2\mathbb{Z}+1$, infinitesimal character $0 < \nu < 1$:
$\langle\ ,\ \rangle$ has opposite signs on the two lowest K-types ± 1.

Recall:

$$\langle \pi(X)v, w\rangle + \langle v, \pi(X)w\rangle = 0 \quad (X \in \mathfrak{g}_0)$$

Want $X \in \mathfrak{g}$:

$$\langle \pi(X)v, w\rangle + \langle v, \pi(\sigma(X))w\rangle = 0 \quad (X \in \mathfrak{g})$$

where σ is anti-holomorphic, and $\mathfrak{g}_0 = \mathfrak{g}^\sigma$.

The c-form

So: while in small examples it is possible to compute these w_i, it is not possible to formulate a general algorithm for them.

Here is the key idea to resolve this: use a modification of the invariant form.

Definition

A c-form on (π, V) is a Hermitian form satisfying:

$$\langle \pi(X)v, w\rangle_c + \langle v, \pi(\sigma_c(X))w\rangle_c = 0 \quad (X \in \mathfrak{g})$$

Here σ_c is the compact real form of $\mathfrak{g}$: $\sigma_c\sigma = \sigma\sigma_c$, and $G(\mathbb{C})^{\sigma_c}$ is the compact form of $G(\mathbb{C})$.

Definition

$sig^c_\pi(\mu) = p + qs$ according to the signature (p, q) of the c-form on the μ-isotypic subspace.

The c-form

Recall it isn't possible formulate a precise algorithm to compute sig_π.

Theorem (Atlas - Life is beautiful)

Suppose (π, V) is irreducible, with real infinitesimal character.

- $\langle\, , \rangle_c$ *exists*
- $\langle\, , \rangle_c$ *is canonical (positive on all lowest K-types)*
- *There is an explicit formula relating $\langle\, , \rangle_c$ and $\langle\, , \rangle$.*

So it is at least possible to formulate an algorithm to compute sig^c_π, and therefore sig_π.

We want a formula for

$$\mathrm{sig}^c_\pi = \sum_{i=1}^{n} w^c_i \mathrm{mult}_{\pi_i} \quad (\pi_i \text{ tempered})$$

and a way to go from sig^c_π to sig_π.

The ATLAS Algorithm: Main Step

Each irreducible representation π is the unique irreducible quotient of a standard module

$$I(P, \sigma, \nu)$$

$P = MAN$ is a real parabolic subgroup

σ is a discrete series of M

and $\nu \in \mathfrak{a}^* \simeq \mathbb{C}^n$

Real infinitesimal character: $\nu \in \mathbb{R}^n$.

Tempered: $\nu = 0$.

Key step: Deformation

$I(\nu) = I(P, \sigma, \nu)$: all realized on the same vector space V.

Write the Hermitian form $\langle\, , \rangle_\nu$ on $I(\nu)$.

Deform ν to 0: $\langle\, , \rangle_\nu$ only changes at (finitely many) reducibility points.

At a reducibility point ν_0 we have the Jantzen filtration.

Basic principle:

$\langle\, , \rangle_{\nu_0+\epsilon}$ and $\langle\, , \rangle_{\nu_0-\epsilon}$ differ on odd levels of the Jantzen filtration.

(Essentially: $f(x) = x^n$ changes sign at 0 if and only if n is odd.)

So: deform ν to 0; keep track of the sign changes

At $\nu = 0$ the representation is tempered, and all signs are positive.

ATLAS Algorithm for the signature of the c-form

Deformation (see Astérisque/arXiv):

$$\mathrm{sig}^c(I((1+\epsilon)\nu)) = \mathrm{sig}^c(I((1-\epsilon)\nu) + (1-s) \sum_{\substack{\phi,\tau \\ \phi<\tau<\gamma \\ \ell(\gamma)-\ell(\tau)\ \mathrm{odd}}} s^{(\ell_0(\gamma)-\ell_0(\tau))/2} P_{\phi,\tau}(s) Q_{\tau,\gamma}(s) \mathrm{sig}^c(\mathrm{I}(\phi))$$

P, Q : the Kazhdan-Lusztig-Vogan polynomials $\in \mathbb{Z}[q]$.

Applied recursively: $\nu \to 0$ (tempered), this gives a computable formula

$$\boxed{\mathrm{sig}^c(\mathrm{I}) = \sum_{i=1}^{n} w_i^c \cdot \mathrm{mult}_{\pi_i} \quad (\pi_i \quad \text{tempered})}$$

ATLAS Algorithm for the signature of the c-form

Furthermore: you can express an irreducible representation π (with its form) as a linear combination of standard representations to give a formula for sig_π^c.

Finally the c-form and the ordinary form are related in the equal rank case by:

$$\langle v, w\rangle = \zeta_\pi \langle \pi(x)v, w\rangle_c$$

where ζ_π is a fixed root of unity and $x \in K, x^2 \in Z(G)$.

Most significant thing swept under the rug:in the unequal rank case we need an extended group $K \rtimes \delta$ with $\delta^2 = 1$, and $x \in K\delta$. This requires the twisted Kazhdan-Lusztig-Vogan polynomials studied by Lusztig and Vogan (2014).

Application: Unipotent Representations

Let's consider the most interesting unitary representations.

$\Gamma = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$

G : connected reductive algebraic group, defined over $\mathbb{R}$

${}^\vee G^\Gamma = G^\vee \rtimes \Gamma$

$$\begin{tikzcd} \Psi : SL(2,\mathbb{C}) \times \Gamma \arrow[rr] \arrow[dr] & & {}^\vee G^\Gamma \arrow[dl] \\ & \Gamma & \end{tikzcd}$$

This is a unipotent Arthur parameter

Conjecture (Arthur - 1983)

Associated to Ψ is a finite set $\Pi(\Psi)$ of irreducible "unipotent" representations.

Application: Unipotent Representations

Arthur gave some properties these packets should have, including:

1. $\Pi(\Psi)$ contains a certain L-packet $\Pi(\phi_\psi)$
2. The representations in $\Pi(\psi)$ have infinitesimal character $\lambda_\Psi = d\Psi(\mathrm{diag}(\frac{1}{2}, -\frac{1}{2}))$
3. The span of the distribution characters $\{\theta_\pi \mid \pi \in \Pi(\Psi)\}$ contains a stable character.
4. The representations in $\Pi(\Psi)$ are unitary.

Arthur conjectured that the (global) unipotent representations are the building blocks for general automorphic representations.

Locally: the unipotent representation of real groups are expected to be the building blocks of the unitary dual.

What is a Unipotent Representation/Packet?

Arthur did not give a definition of what unipotent representations should actually be.

The terms unipotent representation/packet have come to mean many different things, over $\mathbb{R}$, and over local and global fields (Barbasch and others), and are not precisely defined in full generality (see Vogan's Orange book for more on this).a

Here we only discuss $\mathbb{R}$, and well defined notions of Arthur unipotent (aka Special unipotent) representations/packets.

These were originally defined by Barbasch and Vogan [1986] for complex groups, and Adams/Barbasch/Vogan [1992] for real groups.

It is natural to ignore the restriction to Γ:

Definition

Suppose $\mathcal{O}^\vee \subset G^\vee$ is a unipotent orbit. The weak Arthur packet associated to $\mathcal{O}^\vee$ is

$$\Pi(\mathcal{O}^\vee) = \bigcup_{\substack{\Psi : SL(2,\mathbb{C}) \times \Gamma \to {}^\vee G^\Gamma \\ \Psi|_{SL(2,\mathbb{C})} \longleftrightarrow \mathcal{O}^\vee}} \Pi(\Psi)$$

It turns out it is easier to describe weak Arthur packets $\Pi(\mathcal{O}^\vee)$.

Weak Arthur packets

We now apply the definition of $\Pi(\Psi)$ [ABV 1992] to compute $\Pi(\mathcal{O}^\vee)$. This becomes essentially [BV 1986] and has the following simple form.

Theorem

The weak Arthur packet $\Pi(\mathcal{O}^\vee)$ consists of the irreducible representations π of G satisfying:

1. *The infinitesimal character of π is $\lambda_{\mathcal{O}^\vee}$*
2. $AV(Ann(\pi)) = \overline{d\mathcal{O}^\vee}$

Ingredients:

d is Spaltenstein/Barbasch/Vogan duality: $\mathrm{Unip}(G^\vee) \to \mathrm{Unip}(G)$

$\mathrm{Ann}(\pi)$ is the annihlator of π in $\mathcal{U}(\mathfrak{g})$

$\mathrm{AV}(\mathrm{Ann}(\pi))$ is the associated variety of $\mathrm{Ann}(\pi)$: the closure of a complex unipotent orbit $\mathcal{O}$ of G.

Unipotent Representations

Definition

The unipotent representations $\Pi_u(G)$ of G are the union of the weak Arthur packets.

$$\Pi_{\mathsf{Unip}}(G) = \bigcup_{\mathcal{O}^\vee \in \mathsf{Unip}(G^\vee)} \Pi(\mathcal{O}^\vee)$$

Remark: Honest Arthur packets has a similar definition, but involving the finer invariant $\mathsf{AV}(\pi)$ in place of $\mathsf{AV}(\mathsf{Ann}(\pi))$.

Unipotent Representations in Atlas

Theorem

There is an explicit algorithm to compute each weak Arthur packet $\Pi(\mathcal{O}^\vee)$, and therefore $\Pi_{Unip}(G)$.

This algorithm has been implement in the `ATLAS` software: given G and $O^\vee$, we compute an explicit list of Langlands parameters of the representations in $\Pi_G(O^\vee)$.

Unipotent Representations in Atlas

Ingredients:

- Unipotent orbits $\{\mathcal{O}^\vee\}$ in $G^\vee$
- $S(\mathcal{O}^\vee) = \mathrm{Cent}_{G^\vee}(\psi)$ (a disconnected reductive group)
- Action of the Weyl group on the Grothendieck group *Gr* of virtual $(\mathfrak{g}, K)$-modules
- The Kazhdan-Lusztig-Vogan polynomials
- The cell decomposition of *Gr*
- Special representation of each cell
- Duality of nilpotent orbits
- Springer correspondence (*)
  (*) Only the Springer correspondence requires tables: everything else is computed *de nuovo* for an arbitrary reductive group.

Unipotent Representations of Exceptional Groups

G	K	#Unip
G_2(cpt)	$G2$	1
G_2(split)	$2A1$	12
F_4(compact)	$F4$	1
F_4 (B_4)	$B4$	3
F_4 (split)	$A1+C3$	75
E_6^{sc} (compact)	$E6$	1
E_6^{sc}(herm.)	$D5+T1$	12
E_6^{sc}(quasisplit)	$A1+A5$	47
E_6^{sc}(F_4)	$F4$	3
E_6^{sc}(split)	$C4$	68
E_6^{ad}(compact)	E_6	1
E_6^{ad}(herm.)	$D5+T1$	12
E_6^{ad}(quasisplit)	$A1+A5$	47
E_6^{ad}(F_4)	$F4$	3
E_6^{ad}(split)	$C4$	68

G	K	#Unip
E_7^{sc}(compact)	E_7	1
E_7^{sc} (herm.)	E_6+T1	28
E_7^{sc}(quat.)	$A1+D6$	56
E_7^{sc}(split)	$A7$	252
E_7^{ad}(compact)	E_7	1
E_7^{ad}(herm.)	E_6+T1	23
E_7^{ad}(quat.)	$A1+D6$	54
E_7^{ad}(split)	$A7$	276
E_8(compact)	E_8	1
E_8(quat.)	$A1+E_7$	57
E_8 (split)	$D8$	362
TOTAL		1,465

Example: G_2

i	diagram	dim	BC Levi	Cent_0	Z	C2	A(O)	#AP	Cent(O)	#reps
0	[0,0]	0	2T1	G2	1	2	[1]	2	G2	2=1+1
1	[1,0]	6	A1+T1	A1	2	2	[1]	2	SL(2)	2=1+1
2	[0,1]	8	A1+T1	A1	2	2	[1]	2	SL(2)	2=1+1
3	[2,0]	10	G2	e	1	1	[1,2,3]	2	S3	5=2+3
4	[2,2]	12	G2	e	1	1	[1]	1	G2	1=1

orbit#	parameters	inf. char.
0	param(x=9,lambda=[1,1]/1,nu=[0,0]/1)	[0,0]
0	param(x=0,lambda=[0,0]/1,nu=[0,0]/1)	[0,0]
1	param(x=9,lambda=[1,1]/1,nu=[1,0]/2)	[1/2,0]
1	param(x=9,lambda=[2,1]/1,nu=[1,0]/2)	[1/2,0]
2	param(x=9,lambda=[1,2]/1,nu=[0,1]/2)	[0,1/2]
2	param(x=9,lambda=[1,1]/1,nu=[0,1]/2)	[0,1/2]
3	param(x=4,lambda=[1,0]/1,nu=[2,-1]/2)	[1,0]
3	param(x=8,lambda=[3,0]/1,nu=[1,0]/1)	[1,0]
3	param(x=2,lambda=[1,0]/1,nu=[0,0]/1)	[1,0]
3	param(x=6,lambda=[4,-1]/1,nu=[3,-1]/2)	[1,0]
3	param(x=9,lambda=[1,1]/1,nu=[1,0]/1)	[1,0]
4	param(x=9,lambda=[1,1]/1,nu=[1,1]/1)	[1,1]

Example: E_8

i	diagram	dim	BC Levi	Cent_0	Z	A(O)	C2	#A	Cent(O)	#Unip	split	quat.
0	[0,0,0,0,0,0,0,0]	0	8T1	E8	1	[1]	3	3	E8		3=1+1+1	0
1	[0,0,0,0,0,0,0,1]	58	A1+7T1	E7	2	[1]	4	4	E7		4=1+1+1+1	0
2	[1,0,0,0,0,0,0,0]	92	2A1+6T1	B6	2	[1]	5	5	Spin(13)		5=1+1+1+1+1	0
3	[0,0,0,0,0,0,1,0]	112	3A1+5T1	A1+F4	2	[1]	6	6	SL(2)xF4		6=1+1+1+1+1+1	0
4	[0,0,0,0,0,0,0,2]	114	A2+6T1	E6	3	[1,2]	3	5	E6|2		10	0
5	[0,1,0,0,0,0,0,0]	128	4A1+4T1	C4	2	[1]	5	5	Sp(8)		5=1+1+1+1+1	0
6	[1,0,0,0,0,0,0,1]	136	A1+A2+5T1	A5	6	[1,2]	4	6	SL(6)		12	0
7	[0,0,0,0,0,1,0,0]	146	2A1+A2+4T1	A1+B3	2	[1]	5	5	SL(2)xSpin(7)/(-1,-1)		6	0
8	[1,0,0,0,0,0,0,2]	148	A3+5T1	B5	2	[1]	4	4	Spin(11)		4=1+1+1+1	0
9	[0,0,1,0,0,0,0,0]	154	3A1+A2+3T1	A1+G2	2	[1]	4	4	SL(2)xG2		4=1+1+1+1	0
10	[2,0,0,0,0,0,0,0]	156	2A2+4T1	2G2	1	[1,2]	4	4	[2G2]|2		7	0
11	[1,0,0,0,0,0,1,0]	162	A1+2A2+3T1	A1+G2	2	[1]	4	4	SL(2)xG2		4=1+1+1+1	0
12	[0,0,0,0,0,1,0,1]	164	A1+A3+4T1	A1+B3	4	[1]	6	6	SL(2)xSpin(7)		6=1+1+1+1+1+1	0
13	[0,0,0,0,0,0,2,0]	166	D4+4T1	D4	4	[1,2,3]	5	5	Spin(8)|S3		12	0
14	[0,0,0,0,0,0,2,2]	168	D4+4T1	F4	1	[1]	3	3	F4		3	2
15	[0,0,0,0,1,0,0,0]	168	2A1+2A2+2T1	B2	2	[1]	3	3	Spin(5)		3	0
16	[0,0,1,0,0,0,0,1]	172	2A1+A3+3T1	A1+B2	4	[1]	6	6	SL(2)xSpin(5)		6	0
17	[0,1,0,0,0,0,1,0]	176	A1+D4+3T1	3A1	8	[1,2,3]	8	6	[SL(2)^3]|S3		14	0
18	[1,0,0,0,0,1,0,0]	178	A2+A3+3T1	B2+T1	2	[1,2]	4	6	GSpin(5)|2		8	0
19	[2,0,0,0,0,0,0,2]	180	A4+4T1	A4	5	[1,2]	3	4	SL(5).2		8	0
20	[0,0,0,1,0,0,0,0]	182	A1+A2+A3+2T1	2A1	2	[1]	4	4	SL(2)xSL(2)		4=1+1+1+1	0
21	[0,1,0,0,0,0,1,2]	184	A1+D4+3T1	C3	2	[1]	4	4	Sp(6)		4=1+1+1+1	2
22	[0,2,0,0,0,0,0,0]	184	A2+D4+2T1	A2	1	[1,2]	2	3	PSL(3)|2		6	0
23	[1,0,0,0,0,1,0,1]	188	A1+A4+3T1	A2+T1	3	[1,2]	4	4	GL(3).2		8	0
24	[1,0,0,0,1,0,0,0]	188	2A3+2T1	C2	2	[1]	3	3	Sp(4)		3=1+1+1	0
25	[1,0,0,0,0,1,0,2]	190	D5+3T1	A3	4	[1,2]	3	5	SL(4)|2		10	0
26	[0,0,0,1,0,0,0,1]	192	2A1+A4+2T1	A1+T1	2	[1,2]	3	4	GL(2).2		8	0
27	[0,0,0,0,0,2,0,0]	194	A2+A4+2T1	2A1	2	[1]	3	3	SL(2)xSL(2)/<-1,-1>		5	0
28	[2,0,0,0,0,1,0,1]	196	A5+3T1	A1+G2	2	[1]	4	4	SL(2)xG2		4=1+1+1+1	0
29	[0,0,0,1,0,0,0,2]	196	A1+D5+2T1	2A1	2	[1]	4	4	SL(2)xPSL(2)		5	0
30	[0,0,1,0,0,1,0,0]	196	A1+A2+A4+T1	A1	2	[1]	2	2	SL(2)		2=1+1	0
31	[2,0,0,0,0,0,2,0]	198	E6+2T1	G2	1	[1,2]	2	4	G2xZ2		8	0
32	[0,2,0,0,0,0,0,2]	198	A2+D4+2T1	A2	3	[1,2]	2	3	SL(3)|2		4	1

i	diagram	dim	BC Levi	Cent_0	Z	A(O)	C2	#A	Cent(O)	#Unip split	quat.
33	[2,0,0,0,0,0,2,2]	200	D5+3T1	B3	2	[1]	3	3	Spin(7)	3=1+1+1	3
34	[0,0,0,1,0,0,1,0]	200	A3+A4+T1	A1	2	[1]	2	2	SL(2)	2=1+1	0
35	[1,0,0,1,0,0,0,1]	202	A1+A5+2T1	2A1	4	[1]	2	4	SL(2)xSL(2)	4=1+1+1+1	0
36	[0,0,1,0,0,1,0,1]	202	A2+D5+T1	A1	2	[1]	2	2	SL(2)	2=1+1	0
37	[0,1,1,0,0,0,1,0]	204	D6+2T1	2A1	4	[1,2]	4	4	[SL(2)xSL(2)].2	7	0
38	[1,0,0,0,1,0,1,0]	204	A1+E6+T1	A1	2	[1,2]	2	4	SL(2)xZ2	8	0
39	[0,0,0,1,0,1,0,0]	206	E7+T1	A1	2	[1,2,3]	2	4	SL(2)xS3	10	0
40	[1,0,0,0,1,0,1,2]	208	A1+D5+2T1	2A1	4	[1]	4	4	SL(2)xSL(2)	4=1+1+1+1	2
41	[0,0,0,0,2,0,0,0]	208	E8	e	1	[S5]	1	3	S5	18=7+6+5	0
42	[2,0,0,0,0,2,0,0]	210	A6+2T1	2A1	2	[1]	3	3	SL(2)xSL(2)/<-1,-1>	5	0
43	[0,1,1,0,0,0,1,2]	210	D6+2T1	2A1	4	[1,2]	4	4	[SL(2)xSL(2)].2	7	4
44	[0,0,0,1,0,1,0,2]	212	E7+T1	A1	2	[1,2]	2	4	SL(2)xZ2	5	0
45	[1,0,0,1,0,1,0,0]	212	A1+A6+T1	A1	2	[1]	2	2	SL(2)	2=1+1	0
46	[2,0,0,0,0,2,0,2]	214	E6+2T1	A2	3	[1,2]	2	3	SL(3)|2	6	1
47	[0,0,0,0,2,0,0,2]	214	A2+D5+T1	T1	1	[1,2]	2	3	GL(1)|2	4	1
48	[2,0,0,0,0,2,2,2]	216	E6+2T1	G2	1	[1]	2	2	G2	2=1+1	3
49	[2,1,1,0,0,0,1,2]	216	D6+2T1	B2	2	[1]	3	3	Spin(5)	3=1+1+1	3
50	[1,0,0,1,0,1,0,1]	216	D7+T1	T1	1	[1,2]	2	2	GL(1).2	4	0
51	[1,0,0,1,0,1,0,2]	218	A1+E6+T1	T1	1	[1,2]	2	2	GL(1).2	4	0
52	[1,0,0,1,0,1,1,0]	218	A7+T1	A1	2	[1]	2	2	SL(2)	2=1+1	0
53	[2,0,0,1,0,1,0,2]	220	E7+T1	A1	2	[1,2]	2	4	SL(2)xZ2	8	2
54	[0,0,0,2,0,0,0,2]	220	E8	e	1	[1,2,3]	1	2	S3	6	0
55	[1,0,0,1,0,1,2,2]	222	A1+E6+T1	A1	2	[1]	2	2	SL(2)	2=1+1	2
56	[2,0,0,0,2,0,0,2]	222	D7+T1	T1	1	[1,2]	2	3	T|2	5	1
57	[0,1,1,0,1,0,2,2]	224	E7+T1	A1	2	[1]	2	2	SL(2)	2=1+1	3
58	[0,0,0,2,0,0,2,0]	224	E8	e	1	[1,2,3]	1	2	S3	5	0
59	[2,1,1,0,1,1,0,1]	226	D7+T1	A1	2	[1]	2	2	SL(2)	2=1+1	1
60	[0,0,0,2,0,0,2,2]	226	E8	e	1	[1,2,3]	1	2	S3	5	7
61	[2,1,1,0,1,0,2,2]	228	E7+T1	A1	2	[1]	2	2	SL(2)	2=1+1	1
62	[2,0,0,2,0,0,2,0]	228	E8	e	1	[1,2]	1	2	Z2	6	2
63	[2,0,0,2,0,0,2,2]	230	E8	e	1	[1,2]	1	2	Z2	3	2
64	[2,1,1,0,1,2,2,2]	232	E7+T1	A1	2	[1]	2	2	SL(2)	2=1+1	2
65	[2,0,0,2,0,2,0,2]	232	E8	e	1	[1,2]	1	2	Z2	4	1
66	[2,0,0,2,0,2,2,2]	234	E8	e	1	[1,2]	1	2	Z2	4	6
67	[2,2,2,0,2,0,2,2]	236	E8	e	1	[1]	1	1	1	1=1	2
68	[2,2,2,0,2,2,2,2]	238	E8	e	1	[1]	1	1	1	1=1	1
69	[2,2,2,2,2,2,2,2]	240	E8	e	1	[1]	1	1	1	1=1	1

Example: The orbit $E_8(a7)$

```
i  diagram            dim  BC Levi Cent_0 Z  A(O)    C2 #A Cent(O)    #Unip
41 [0,0,0,0,2,0,0,0]  208  E8      e      1  [S5]    1  3  S5         18=7+6+5


One weak Arthur packet, disjoint union of three (honest) Arthur packets:

\Psi(j)=e, \Cent(\Psi(j))=S_5, |{S_5}^|=7
parameter(G,114501,[0,0,-4,-4,13,0,-4,0]/1,[0,0,-1,-1,3,0,-1,0]/1)
parameter(G,216857,[-4,0,0,-2,14,-7,0,-2]/1,[-2,0,0,-1,6,-3,0,-1]/2)
parameter(G,268640,[0,-5,-2,0,12,-2,-2,0]/1,[0,-2,-1,0,5,-1,-1,0]/2)
parameter(G,289903,[0,-4,0,0,9,1,-5,0]/1,[0,-1,0,0,2,0,-1,0]/1)
parameter(G,304971,[-4,1,1,-5,15,1,-3,-1]/1,[-1,0,0,-1,4,0,-1,0]/2)
parameter(G,316982,[3,3,-6,1,8,1,1,-5]/1,[0,0,-1,0,3,0,0,-1]/2)
parameter(G,320205,[1,1,1,1,1,1,1,1]/1,[0,0,0,0,1,0,0,0]/1)   (spherical)

\Psi(j)=(1,2) \Cent(\Psi(j))=S_2xS_3 |(S_2xS_3)^|=6
parameter(G,206741,[-4,0,-2,-2,13,-4,0,-2]/1,[-2,0,-1,-1,6,-2,0,-1]/2)
parameter(G,257336,[-6,-4,-1,0,14,-1,-3,-2]/1,[-1,-1,-1,0,5,-1,-1,-1]/2)
parameter(G,287349,[0,-4,0,0,10,0,-5,0]/1,[0,-1,0,0,2,0,-1,0]/1)
parameter(G,293421,[0,-2,-2,0,9,1,-3,-2]/1,[0,-1,-1,0,4,0,-1,-1]/2)
parameter(G,309039,[0,-2,-3,2,9,-2,2,-4]/1,[0,0,-1,0,3,0,0,-1]/2)
parameter(G,318032,[0,0,1,-1,6,1,1,-2]/1,[0,0,0,0,1,0,0,0]/1)

\Psi(j)=(1,2)(3,4) \Cent(\Psi(j))=D_4 |D_4^|=5
parameter(G,212118,[-2,-2,-2,-2,13,-2,-2,-2]/1,[-1,-1,-1,-1,6,-1,-1,-1]/2)
parameter(G,278536,[-4,-1,1,-4,13,0,-1,-3]/1,[-1,-1,0,-1,5,-1,0,-1]/2)
parameter(G,289899,[0,-2,-2,0,10,0,-3,-2]/1,[0,-1,-1,0,4,0,-1,-1]/2)
parameter(G,299063,[-3,0,0,-2,11,0,-3,0]/1,[-1,0,0,-1,4,0,-1,0]/2)
parameter(G,317576,[0,0,1,-1,6,1,0,-1]/1,[0,0,0,0,1,0,0,0]/1)
```

Unitarity

Every unipotent representation of $\mathrm{GL}(n,\mathbb{R})$ or $\mathrm{GL}(n,\mathbb{C})$ is unitary.

If $G = SO(n)$ or $Sp(2n)$ (split) Jim Arthur has shown that every unipotent representation is unitary (any local field).

Theorem (Atlas-Stephen D. Miller/Joe Hundley)

Every unipotent representation of a real exceptional group is unitary, with the possible exception of 4 representations of E_8 (split).

Unitarity

Sketch:

Miller and Hundley have shown that some of the unipotent representations of split groups can be realized as L^2-residues of Eisenstein series, and are therefore unitary.

If $\mathcal{O}^\vee$ is not distinguished (it intersects a proper Levi subgroup $L^\vee$), then $\Pi_G(\mathcal{O}^\vee)$ can be obtained by unitary (or cohomological) induction from $\Pi_L(\mathcal{O}_L^\vee)$ (L is a real (or θ-stable) Levi factor of G).

The `ATLAS` software can determine if each π is unitary. However: for some representations of E_8 this is not possible due to time and/or memory limitations.

The unitarity of many unipotent representations can be checked by several of these methods.

Conjecture I

Each Arthur packet comes with a map

$$\tau : \Pi(\Psi) \to \mathcal{R}(\mathcal{S}_\Psi)$$

where ($\mathcal{S}_\Psi$ is the component group of $\mathrm{Cent}_{G^\vee}(\Psi)$):

$$\mathcal{S}_\Psi = S_\Psi / S_\Psi^0$$

$\mathcal{R}$ is the finite dimensional representations (not necessarily irreducible)

Conjecture

For Ψ a unipotent parameter τ is a bijection

$$\Pi(\Psi) \xleftrightarrow{1-1} \widehat{\mathcal{S}_\Psi}$$

This is known to be false for more general (non-unipotent) Arthur packets

We know of no counterexamples to the Conjecture.

Conjecture II: Tempered Unipotent Arthur Packets

Question: When does $\Pi(\mathcal{O}^\vee)$ consist entirely of tempered representations?

Very roughly: never, since the whole point of the $\mathrm{SL}(2,\mathbb{C})$ factor is to account for non-tempered representations.

Roughly: If $\Psi|_{\mathrm{SL}(2,\mathbb{C})} = 1$

Proposition

1) If G is split then $\Pi(\mathcal{O}^\vee)$ is nonempty for all $\mathcal{O}^\vee$.

2) Let $\mathcal{O}^\vee = \{0\}$. Then $\Pi(\mathcal{O}^\vee)$ is nonempty if and only if G is quasisplit. In this case

$$\Pi(\Psi) = \{\pi \mid \textit{infinitesimal character of } \pi = 0\}$$

and these representations are all tempered.

(See the talk by Lucas Mason-Brown, in this seminar on May 29, for more on 2).

Conjecture II: Tempered Unipotent Arthur Packets

Conjecture

$\Pi(\mathcal{O}^\vee)$ consists entirely of tempered representations if and only if $\mathcal{O}^\vee$ is the minimal orbit for G such that $\Pi(\mathcal{O}^\vee)$ is nonempty.

This is true for G quasisplit (previous Proposition), or compact (duh),and for all real forms of exceptional groups (case-by-case).

Example: quaternionic form of E_8

i	diagram	dim	BC Levi	Cent_0	Z	A(O)	C2	#A	Cent(O)	#Unip	split	quat.
0	[0,0,0,0,0,0,0,0]	0	8T1	E8	1	[1]	3	3	E8		3=1+1+1	0
1	[0,0,0,0,0,0,0,1]	58	A1+7T1	E7	2	[1]	4	4	E7		4=1+1+1+1	0
2	[1,0,0,0,0,0,0,0]	92	2A1+6T1	B6	2	[1]	5	5	Spin(13)		5=1+1+1+1+1	0
3	[0,0,0,0,0,0,1,0]	112	3A1+5T1	A1+F4	2	[1]	6	6	SL(2)xF4		6=1+1+1+1+1+1	0
4	[0,0,0,0,0,0,0,2]	114	A2+6T1	E6	3	[1,2]	3	5	E6|2		10	0
5	[0,1,0,0,0,0,0,0]	128	4A1+4T1	C4	2	[1]	5	5	Sp(8)		5=1+1+1+1+1	0
6	[1,0,0,0,0,0,0,1]	136	A1+A2+5T1	A5	6	[1,2]	4	6	SL(6)		12	0
7	[0,0,0,0,0,1,0,0]	146	2A1+A2+4T1	A1+B3	2	[1]	5	5	SL(2)xSpin(7)/(-1,-1)		6	0
8	[1,0,0,0,0,0,0,2]	148	A3+5T1	B5	2	[1]	4	4	Spin(11)		4=1+1+1+1	0
9	[0,0,1,0,0,0,0,0]	154	3A1+A2+3T1	A1+G2	2	[1]	4	4	SL(2)xG2		4=1+1+1+1	0
10	[2,0,0,0,0,0,0,0]	156	2A2+4T1	2G2	1	[1,2]	4	4	[2G2]|2		7	0
11	[1,0,0,0,0,0,1,0]	162	A1+2A2+3T1	A1+G2	2	[1]	4	4	SL(2)xG2		4=1+1+1+1	0
12	[0,0,0,0,0,1,0,1]	164	A1+A3+4T1	A1+B3	4	[1]	6	6	SL(2)xSpin(7)		6=1+1+1+1+1+1	0
13	[0,0,0,0,0,0,2,0]	166	D4+4T1	D4	4	[1,2,3]	5	5	Spin(8)|S3		12	0
14	[0,0,0,0,0,0,2,2]	168	D4+4T1	F4	1	[1]	3	3	F4		3	2

Orbit 14:

```
14: parameter(G,65,[0,0,0,0,0,0,1,1]/1,[0,0,0,0,0,0,0,0]/1),
14: parameter(G,4101,[-3,1,1,1,1,-3,1,1]/1,[0,0,0,0,0,0,0,0]/1)
```

Thank you David!