Lie algebras and unstable homotopy theory

Goal: Understand the htpy theory of spaces.

Problem: Difficult!

Solution: Consider a restricted class of spaces only containing one level of information.

Def: We say a pointed, simply-connected space X is *rational* if $\pi_k X$ is a $\mathbb{Q}$-v.s. for each $k \geq 2$.

$$\begin{tikzcd} \mathcal{S}_*^{\geq 2} \arrow[r, bend left, "\exists!\, L_{\mathbb{Q}}"] & \mathcal{S}_{\mathbb{Q}} \arrow[l, bend left, "\text{inclusion}"] \end{tikzcd}$$

universal functor inverting $H_*(-,\mathbb{Q})$-iso.

Thm. (Quillen)

$$\mathcal{S}_{\mathbb{Q}}^{\geq 2} \xrightarrow[\text{reduced rational cochains}]{\tilde{C}_*(-,\mathbb{Q})} \mathrm{coAlg}^{nu}(Sp_{\mathbb{Q}}^{\geq 2}) \xrightarrow[\text{Chevalley-Eilenberg complex}]{CE} s\mathrm{Lie}(Sp_{\mathbb{Q}}^{\geq 1})$$

are equivalences of ∞-categories.

Q: Can we generalize Quillen's result beyond rational phenomena?

$S_{\mathbb{Q}}$ is the localization of $S_*^{\geq 2}$ at the class of maps $X \to Y$ s.t. p-local spaces

$$p^{-1}\pi_* X \to p^{-1}\pi_* Y$$

is an iso. Here, $p: S^k \to S^k$ is an example of a v_0-map, i.e. one inducing iso. on $K(0)_* = H_*(-;\mathbb{Q})$ and zero on $K(n)_*$ for $n > 0$.

Idea: Replace S^k by other finite complexes and their v_i-self maps.

Recall: If V is a finite, p-local pointed space then its type is the smallest n such that

$$\widetilde{K(n)}_*(V) \neq 0.$$

reduced Morava K-theory homology

(After possibly finitely many suspensions), any type n cpx admits a v_n-self map

$$\Sigma^t V \to V$$

which is iso. on $\widehat{K(n)}_*$ and zero on $\widehat{K(m)}_*$ for $m \neq n$.

Note: 1) If $n > 0$, then $t > 0$.

2) $\mathrm{cof}(v_n : \Sigma^t V \to V)$ is of type $n+1$.

Example: $S^k/p = \mathrm{cof}(p : S^k \to S^k)$

$\exists$ a map $v_1 : \Sigma^t S^k/p \to S^k/p$ iso. on KU_* due to Adams.

From now on, let's fix V of type $n > 0$ and a v_n-self map.

Def: If X is a pointed, simply connected space, then the <u>v_n-inverted homotopy</u> groups are given by

$$v_n^{-1}\pi_*(X;V) := \varinjlim \left([V,X] \to [\Sigma^t V, X] \to [\Sigma^{2t} V, X] \to \dots \right)$$

The ∞-category S_{v_n} is the localization of $S_*^{\geq 1}$ at $v_n^{-1}\pi_*(X;V)$-isomorphism.

This means we have a functor $S_*^{\geq 1} \to S_{v_n}$, initial among all functors inverting $v_n^{-1}\pi_*(-;V)$.

Q: Can we describe the ∞-category S_{v_n}?

Note: $v_n^{-1}\pi_*(X;V)$ are the homotopy groups of the infinite loop space

$$\Phi_V(X) = \varinjlim \left(\mathrm{Map}_*(V,X) \to \mathrm{Map}(\Sigma^t V, X) \to \dots \right)$$

Thus, we can identify $\Phi_V(X)$ with (a 0-th space of) a connective spectrum.

The spectrum $\Phi_V(X)$ depends on V, but the insight of Bousfield and Kuhn is that this dependence is somewhat mild.

Notation: $T(n) = v_n^{-1}\Sigma^\infty V \in Sp$.

Thm (Bousfield-Kuhn)
There exists a unique functor

$$\Phi : S_* \longrightarrow Sp_{T(n)}$$

$T(n)$-local spectra, i.e. the stable analogue of S_{v_n}.

such that

$$F(\Sigma^\infty V, \Phi(X))_{\geq 0} \simeq \Phi_V(X)$$

functorially in V of type at least n.

Idea (sketch): Write $S^0_{T(n)} \simeq \varinjlim V_\alpha$ as a colimit of type n spectra and define

$$\Phi(X) = \varprojlim \left(\Phi_V(X)\right)_{T(n)}$$

Note: Since $[V,-]$ detects equivalences of $T(n)$-local spectra, we get a unique factorization

$$\begin{tikzcd} S_* \arrow[rr, "\Phi"] \arrow[dr] & & Sp_{T(n)} \\ & S_{v_n} \arrow[ur, "\Phi"'] & \end{tikzcd}$$

Prop: The functor $\Phi : S_{v_n} \to Sp_{T(n)}$ is a right adjoint.

Idea: It's enough to verify that each of $\Phi_V : S_{v_n} \to Sp_{T(n)}$ is a right adjoint (as their limit). For the Bousfield-Kuhn functor is these the existence of the left adjoint can be verified directly (but it's involved.)

We will write $\Theta : Sp_{T(n)} \to S_{v_n}$ for the left adjoint to the Bousfield-Kuhn functor.

We are finally able to give a Lie-theoretic description of the unstable v_n-periodic htpy theory.

Thm. (Eldred–Heuts–Mathew–Meier, Heuts)

1) The adjunction $\Theta \dashv \Phi : Sp_{T(n)} \rightleftarrows S_{v_n}$ is monadic; i.e. it induces an equivalence of ∞-categories

$$\mathrm{Mod}_{\Phi\Theta}(Sp_{T(n)}) \simeq S_{v_n}$$

modules over $\Phi\Theta$ the monad v_n-periodic spaces

2) There is an equivalence of monads

$$\Phi\Theta \simeq \mathrm{Lie}$$

free spectral Lie algebra

on $T(n)$-local spectra and hence

$$S_{v_n} \simeq \mathrm{Lie}(Sp_{T(n)})$$

$T(n)$-local spectral Lie algebras

Note: The free spectral Lie algebra is given by

$$\mathrm{Lie}(X) = \bigoplus_{k \geq 0} (\partial_k \mathrm{id} \otimes X^{\otimes k})_{\Sigma_k}$$

htpy orbits

We also have the free spectral partition Lie algebra, which in reasonable situations is given by

$$\mathrm{Lie}^{\pi}(X) = \bigoplus_{k\geq 0} (\partial_k \mathrm{Id} \otimes X^{\otimes k})^{\Sigma_k}$$ htpy fixed points.

In the $T(n)$-local context, these two expressions coincide as a consequence of ambidexterity, so that Lie algebras and partition Lie algebras are the same.

Sketch of the proof of Heuts' theorem:

1) Show $\Phi : S_{v_n} \to Sp_{T(n)}$ is conservative (definitional) and preserves sifted colimits (reduce to Φ_V and use $\mathrm{Map}_*(V,-) : S_* \to S_*$ preserves geometric realizations of highly connected spaces). Barr-Beck-Lurie $\Rightarrow \Theta \dashv \Phi$ is monadic

2) Show that <u>any</u> sifted-colimit preserving endofunctor $F : Sp_{T(n)} \to Sp_{T(n)}$ is coanalytic, that is,

$$F(X) \simeq \bigoplus_{k\geq 0} (\partial_k F \otimes X^{\otimes k})_{\Sigma_k}$$

This is somewhat of a miracle, and uses

$$L_{T(n)}\Sigma^\infty : S_{v_n} \to Sp_{T(n)}$$

and its right adjoint, which define a coanalytic comonad on $Sp_{T(n)}$.

$$\Rightarrow \Phi\Theta(X) \simeq \bigoplus_{k\geq 0}\left(\partial_k(\Phi\Theta)\otimes X^{\otimes k}\right)_{\Sigma k}$$

3) Verify that

$$\partial_k\Phi\Theta \simeq L_{T(n)}\partial_k \mathrm{id}_S$$

This uses

$$P_k(\Phi\Theta)(X) \simeq \Phi P_k \mathrm{id}_{S_{v_n}}(\Theta X)$$

and the calculation of the Goodwillie tower of S_{v_n} using that it's a localization of spaces.