Power operations on spectral Lie algebras

6-4-2022

Everything localized at $p=2$, all (co)homology are w/ $\mathbb{F}_2$-coeffs.

$\boxed{1}$ Construction of power operations $\overline{Q}^i$

$$\begin{tikzcd} Sp \arrow[r, shift left, "Free = H\mathbb{F}_2 \otimes -"] \arrow[r, phantom, "\perp"] & Mod_{H\mathbb{F}_2} \arrow[l, shift left, "U"] \arrow[r, shift left, "Free_{\Phi}"] \arrow[r, phantom, "\perp"] & Alg_{\Phi}(Mod_{H\mathbb{F}_2}) \arrow[l, shift left, "U"] \end{tikzcd}$$

$\mathbb{E}_\infty$-operad: $\Phi = (0, \mathbb{S}, \mathbb{S}, \mathbb{S}, \dots)$ $\quad \Phi = (0, H\mathbb{F}_2, H\mathbb{F}_2, \dots)$

Σ_n @ $\mathbb{S}$ trivially

Want to understand homology ops (for $\mathbb{E}_\infty$-algs)

ie. nat. trans $H_m \to H_n$ of functors $Alg_\Phi \to Mod_{\mathbb{F}_2}$

$\Leftrightarrow$ nat. trans $\pi_m \to \pi_n$ of functors $Alg_\Phi(Mod_{H\mathbb{F}_2}) \to Mod_{\mathbb{F}_2}$.

For $A \in Alg_\Phi(Mod_{H\mathbb{F}_2})$:

$$\pi_m(A) = Map_{Sp}(S^m, A) \simeq Map_{H\mathbb{F}_2}(H\mathbb{F}_2 \otimes S^m, A)$$
$$\simeq Map_{Alg_\Phi(Mod_{H\mathbb{F}_2})}(Free_\Phi(H\mathbb{F}_2 \otimes S^m), A)$$

$\Rightarrow \pi_m$ is rep. by $Free_\Phi(H\mathbb{F}_2 \otimes S^m)$ (in $Alg_\Phi(Mod_{H\mathbb{F}_2})$)

Yoneda $\Rightarrow \{$nat. trans $\pi_m \to \pi_n\} \simeq \pi_n\left(Free_\Phi(H\mathbb{F}_2 \otimes S^m)\right)$

$$\|$$

$Free_{\mathcal{O}}(X) = \bigoplus_k \left(\mathcal{O}(k) \otimes X^{\otimes k}\right)_{h\Sigma_k}$ $\qquad \bigoplus_k \pi_n\left(\left(H\mathbb{F}_2 \otimes (H\mathbb{F}_2 \otimes S^m)^{\otimes k}\right)_{h\Sigma_k}\right)$

$$\cup\!\!|$$

weight 2 operations $\longleftrightarrow \pi_n\left(\left(H\mathbb{F}_2 \otimes (H\mathbb{F}_2 \otimes S^m)^{\otimes 2}\right)_{h\Sigma_2}\right)$

$$\begin{gathered} \| \\ \pi_n \left(H\mathbb{F}_2 \otimes (\mathbb{S}^{2m})_{h\Sigma_2}\right) \\ \| \\ H_n(\Sigma^{2m} \underbrace{\mathbb{RP}^\infty}_{B\Sigma_{2+}}) = \begin{cases} \mathbb{F}_2 & n \geq 2m \\ 0 & \text{o/w} \end{cases} \end{gathered}$$

$\Rightarrow \exists!$ (non triv) Dyer-Lashof op $\pi_m \xrightarrow{Q^i} \pi_{m+i}$ if $m+i \geq 2m$ i.e. $i \geq m$

$s\mathcal{L} := \partial_*(id) = (0, \partial_1(id), \partial_2(id), \partial_3(id), \ldots)$ spectral Lie operad

Recall (Natalia's talk) :

$$\partial_n(id) = \underset{\text{SW dual}}{D}(\Sigma^\infty \overset{\text{Partition cpx}}{P_n}) = \mathrm{Map}_*(\Sigma^\infty P_n, \mathbb{S}) \quad \text{(Arone-Mahowald)}$$

$\partial_1(id) = \mathbb{S}^0$, $\partial_2(id) = \mathbb{S}^{-1} \circlearrowleft \Sigma_2$ triv

Want : Homology ops $\pi_m \longrightarrow \pi_n$ as funcs $\mathrm{Alg}_{s\mathcal{L}}(\mathrm{Mod}_{H\mathbb{F}_2}) \longrightarrow \mathrm{Mod}_{\mathbb{F}_2}$

$$\{\text{nat. trans } \pi_m \to \pi_m\} \simeq \pi_n\left(\mathrm{Free}_{s\mathcal{L}}(H\mathbb{F}_2 \otimes \mathbb{S}^m)\right)$$

$$\text{weight 2 ops} \longleftrightarrow \begin{gathered} \cup \text{I} \\ H_n(\mathbb{S}^{-1} \otimes (\mathbb{S}^{2m})_{h\Sigma_2}) \end{gathered}$$

$\Rightarrow \exists!$ (nontriv) power op $\pi_m \xrightarrow{\bar{Q}^i} \pi_{m+i-1}$ if $i \geq m$

Dyer-Lashof op for $\mathbb{E}_\infty$-algs $A \in \mathrm{Alg}_{\Phi}(\mathrm{Mod}_{H\mathbb{F}_2})$

$$Q^i : H_n(A) \longrightarrow H_{n+i}(A^{\otimes 2}) \longrightarrow H_{n+i}(A^{\otimes 2}_{h\Sigma_2}) \xrightarrow{\text{struct map}} H_{n+i}(A)$$

$(i \geq n)$

$$x \longmapsto e_{i-n} \otimes x \otimes x \longmapsto e_{i-n} \otimes x \otimes x \longmapsto Q^i(x)$$

$A^{\otimes 2}_{h\Sigma_2} = (\mathbb{S}^0 \otimes A^{\otimes 2})_{h\Sigma_2}$

e_{i-n}: cell of dim $i-n$ $\in H_{n+i}(\Sigma^{2n} \mathbb{R}P^\infty) = H_{i-n}(\mathbb{R}P^\infty)$

$$\begin{tikzcd} A \otimes A \arrow[r] \arrow[d] \arrow[dr, phantom, "\circ"] & A \\ (A \otimes A)_{h\Sigma_2} \arrow[ur, "\text{struct map}"'] & \end{tikzcd}$$

Power op for $s\mathcal{L}$-algs $L \in \mathrm{Alg}_{s\mathcal{L}}(\mathrm{Mod}_{H\mathbb{F}_2})$

$$\bar{Q}^i : \underbrace{H_n(L) \to H_{n+i}(L^{\otimes 2}) \to H_{n+i}(L^{\otimes 2}_{h\Sigma_2})}_{\text{"Dyer-Lashof"}} \xrightarrow{\sim} H_{n+i-1}(\mathbb{S}^{-1} \otimes L^{\otimes 2}_{h\Sigma_2})$$

$(i \geq n)$

Σ_2 ${}^{\text{C}}$ triv on $\mathbb{S}^{-1}$

$$= H_{n+i-1}((\mathbb{S}^{-1} \otimes L^{\otimes 2})_{h\Sigma_2})$$

$$\downarrow \text{struct map}$$

$$H_{n+i-1}(L)$$

$\boxed{2}$ Relations

Dyer-Lashof algebra (Behrens pg 5)

$$R = \mathbb{F}_2\{Q^i\} \,/\, \text{Adem rels}$$

$$Q^r Q^s = \sum_t \binom{t+s-r}{2t-r} Q^{r+s-t} Q^t \qquad (r > 2s)$$

$$R_n = R/\sim \quad — \quad Q^J = 0 \text{ if } j_1 < j_2 + \dots + j_k + n \qquad (J = (j_1, \dots, j_k))$$

J admissible if $j_s \leq 2 j_{s+1}$

$\Rightarrow$ can write every monomial in R_n of length k uniquely as LC of admissibles of length k

$$\Rightarrow R_n = \bigoplus_{k \geq 0} \underbrace{R_n(k)}_{\text{length } k}$$

Similarly (Camarena 5.2, Behrens pg 6),

$$\bar{R} = \mathbb{F}_2\{\bar{Q}^j\} /\sim$$

$$\bar{Q}^r \bar{Q}^s = \sum_{k=0}^{r-s-1} \binom{2s-r+1+2k}{k} \bar{Q}^{2s+1+k} \bar{Q}^{r-s-1-k} \qquad (s < r \leq 2s)$$

$$\bar{R}_n = R/\sim \quad — \quad \bar{Q}^J = 0 \text{ if } j_1 < j_2 + \dots + j_k + n$$

J completely unadmissible (CU) if $j_s > 2 j_{s+1}$ $\forall s$

$\Rightarrow$ can write every monomial in $\bar{R}_n$ of length k uniquely as LC of CUs of length k

Why are these all the relations?

Def: M +vely graded $\bar{\mathcal{R}}$-mod is called <u>allowable</u> if

$\forall x \in M$ homog & $j_1 < j_2 + \dots + j_k + |x| \Rightarrow \bar{Q}^J x = 0$

Prop 5.9: Given $L \in \mathrm{Alg}_{sL}(\mathrm{Mod}_{HF_2})$, $H_{>0}(L)$ is an allowable $\bar{\mathcal{R}}$-mod via $\bar{Q}^j$.

Pf: WTS $\bar{Q}^j$ act allowably & satisfy rels in $\bar{\mathcal{R}}$ ($\bar{Q}^r \bar{Q}^s = \dots$).

Use Behrens Thm 1.5.1:

$$\bigoplus_{k \geq 0} H_*(D_{2^k}(S^n)) = \bar{\mathcal{R}}_n\{i_n\} \text{ as } \bar{\mathcal{R}}_n\text{-mods via } \bar{Q}^j,$$

where $\langle i_n \rangle = \tilde{H}_n(S^n) \simeq H_n(D_1(S^n))$

($\Rightarrow$ the $\bar{Q}^j$ satisfy all rels in $\bar{\mathcal{R}}_n$, which are the rels we want for $|x| = n$)

$$\text{Given } x \in H_n(L) \longleftrightarrow x : S^n \xrightarrow{H_*:\ i_n \mapsto x} HF_2 \otimes L \text{ in } Sp$$

$$x : \Sigma^n HF_2 \to HF_2 \otimes L \text{ in } \mathrm{Mod}_{HF_2}$$

$$\tilde{x} : HF_2 \otimes \mathrm{Free}_{sL}(S^n) \to HF_2 \otimes L \text{ in } \mathrm{Alg}_{sL}(\mathrm{Mod}_{HF_2})$$

$\forall R \in \bar{\mathcal{R}}$,

$$\begin{tikzcd}
H_n(\mathrm{Free}_{sL}(S^n)) \arrow[rrr, "H_*(\tilde{x})"] \arrow[d, equal] & & & H_n(L) \arrow[ddd, "R"] \\
\bigoplus_{k \geq 0} H_n(D_{2^k}(S^n)) \arrow[dd, "R"'] & i_n \arrow[r, mapsto] \arrow[d] \arrow[dr, phantom, "\circ"] & x \arrow[d] & \\
& R i_n \arrow[r, mapsto] & Rx & \\
\bigoplus_{k \geq 0} H_{n+?}(\text{---}) \arrow[rrr, "H_*(\tilde{x})"'] & & & H_{n+?}(L)
\end{tikzcd}$$

So $R i_n = 0 \Rightarrow Rx = 0$ in $H_*(L)$ $\square$

Pf summary of Prop 5.9 :

(1) Arone - Mahowald :

$$\underbrace{\bigoplus_{k\geq 0} H_{*}\left(\mathbb{D}_{2k}(S^n)\right)}_{= H_{>0}(\mathrm{Free}_{s\mathcal{L}}(S^n))} = \bar{\mathcal{R}}_n \{ i_n \} \text{ as } \mathbb{F}_2\text{-mods}$$

$$\left(i_n \in H_n(\mathbb{D}_1(S^n)) \simeq \tilde{H}_n(S^n) \right)$$

(2) Behrens Thm 1.5.1 : iso as $\bar{\mathcal{R}}_n$-mods via $\bar{Q}^j$

$\Rightarrow$ rels hold for $\mathrm{Free}_{s\mathcal{L}}(S^n)$

(3) π_n is "maps out of S^n in Sp" $\equiv$ "maps out of $\mathrm{Free}_{s\mathcal{L}}(S^n)$ in $\mathrm{Alg}_{s\mathcal{L}}$"

& $\bar{Q}^j$ natural so (rels hold for $H_*(\mathrm{Free}_{s\mathcal{L}}(S^n))$) $\Rightarrow$ (rels hold for $x \in H_n(L)$)

$\square$

$\Rightarrow$ $\bar{Q}^j$ satisfy these relations

Why are those all the rels between the $\bar{Q}^j$'s ?

$\forall n$ $H_{>0}(\mathrm{Free}_{s\mathcal{L}}(S^n))$ is free $\bar{\mathcal{R}}_n$-mod (Behrens 1.5.1)

$\Rightarrow$ there cannot be more rels

$\boxed{3}$ Shifted Lie bracket (Camarena 5.1)

Everything over $\mathbb{F}_2$.

$$\mathcal{L}ie \xleftrightarrow{\text{Koszul dual}} \mathcal{C}omm$$

$s\mathcal{L}ie = \partial_*(\mathrm{id})$ = operadic suspension of $\mathcal{L}ie$

Lie-alg : $\mathrm{Alg}_{\mathcal{L}ie}$ $\qquad [-,-] : L_i \otimes L_j \longrightarrow L_{i+j}$

- symmetry : $[x,y] = [y,x]$
- Jacobi id : $[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0$

sLie-alg $\qquad [-,-] : L_i \otimes L_j \longrightarrow L_{i+j-1}$

- symmetry / graded comm
- Jacobi

$L \in \mathrm{Alg}_{s\mathcal{L}}(\mathrm{Mod}_{H\mathbb{F}_2})$

$$\mathbb{S}^{-1} \otimes L \otimes L \longrightarrow \mathbb{S}^{-1} \otimes L^{\otimes 2}_{h\Sigma_2} \longrightarrow L \quad \cdots \circledast$$

$\Rightarrow [-,-] : H_i(L) \otimes H_j(L) \longrightarrow H_{i+j-1}(L)$

$\boxed{\text{Prop : } [-,-] \text{ gives a sLie-alg structure on } H_*(L)}$

Pf :

- (graded) comm : $\Sigma_2 \circlearrowright \mathbb{S}^{-1}$ triv $\Rightarrow \circledast$ is a symmetric map $\Rightarrow \checkmark$
- Jacobi : WTS $[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0$

$[x,[y,z]]$: $\partial_2(\mathrm{id}) \otimes (\partial_1(\mathrm{id}) \otimes \partial_2(\mathrm{id})) \xrightarrow{\sim} \partial_3(\mathrm{id})$

$\mathbb{S}^{-2} = \mathbb{S}^{-1} \otimes \mathbb{S}^{0} \otimes \mathbb{S}^{-1}$

induced by $\mathbb{S}^{-2} \otimes L^{\otimes 3} \longrightarrow (\partial_3(\mathrm{id}) \otimes L^{\otimes 3})_{h\Sigma_3} \longrightarrow L$

let $\sigma = (123) \in \Sigma_3$

$\Rightarrow [y,[z,x]]$ induced by $\sigma_* \circ v : \mathbb{S}^{-2} \xrightarrow{v} \partial_3(id) \xrightarrow{\sigma_*} \partial_3(id)$

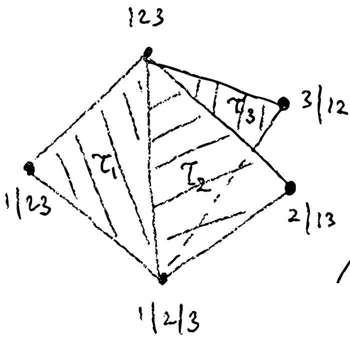
$[z,[x,y]]$ —— $\sigma_*^2 \circ v$

WTS $v + \sigma_* \circ v + \sigma_*^2 \circ v : \mathbb{S}^{-2} \xrightarrow{v} \partial_3(id) \xrightarrow{1+\sigma+\sigma^2} \partial_3(id)$ nullhom

$\partial_3(id) = D(\Sigma^\infty P_3)$

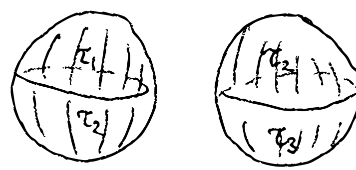
ETS $1+\sigma+\sigma^2 : \Sigma^\infty P_3 \longrightarrow \Sigma^\infty P_3$ nullhom

Recall $\Sigma^\infty P_3$:



$\simeq$

$\Rightarrow \Sigma^\infty P_3 \simeq \underset{\tau_{12}}{S^2} \vee \underset{\tau_{23}}{S^2}$ non-equivariantly



$1+\sigma+\sigma^2 : \Sigma^\infty P_3 \xrightarrow{\Delta} \bigvee^3 \Sigma^\infty P_3 \xrightarrow{1\vee\sigma\vee\sigma^2} \bigvee^3 \Sigma^\infty P_3 \xrightarrow{\nabla} \Sigma^\infty P_3$

$(I \ I \ I)$ — 2×2 matrix

$$\begin{pmatrix} I & & 0 \\ & A & \\ 0 & & A^2 \end{pmatrix} \qquad \begin{pmatrix} I \\ I \\ I \end{pmatrix} = I + A + A^2 = 0$$

A = matrix of $\sigma = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$

$\sigma(\tau_{12}) = \tau_{23}$

$\sigma(\tau_{23}) = \tau_{31} = -\tau_{12} - \tau_{23}$

□

$\boxed{4}$ **Allowable $\bar{R}$-slie-algebra struct on $H_*(L)$:**
compability of $[-,-]$ with $\bar{Q}^j$ (Camarena Section 6)

> **Def:** Allowable $\bar{R}$-slie-algebra = graded $\mathbb{F}_2$-VS M w/
> - shifted Lie bracket
> - $M_{>0}$ is allowable $\bar{R}$-mod
>
> s.t.
>
> (1) $\bar{Q}^{|x|} x = [x, x]$
>
> (2) $[x, \bar{Q}^k y] = 0 \quad \forall x, y$

$L \in Alg_{sL}(Mod_{HF_2})$

> **Thm:** $H_*(L)$ is allowable $\bar{R}$-slie-alg

Pf: Know: $H_*(L)$ is slie-alg & $H_{>0}(L)$ allowable $\bar{R}$-mod
WTS (1) & (2)

$$x \in H_j(L) \longleftrightarrow \Sigma^j HF_2 \xrightarrow{x} HF_2 \otimes L \quad \text{in } Mod_{HF_2}$$

$$\Downarrow$$

$$(-)^{\otimes 2}_{h\Sigma_2} \xrightarrow{x^{\otimes 2}_{h\Sigma_2}} (-)^{\otimes 2}_{h\Sigma_2}$$

$HF_2 \otimes - = Free_{HF_2}(-)$ is sym. monoidal & preserves hocolim (is left adjoint)

$$\Rightarrow (HF_2 \otimes Y)^{\otimes 2}_{h\Sigma_2} \simeq HF_2 \otimes Y^{\otimes 2}_{h\Sigma_2}$$

$$\Rightarrow (\Sigma^j HF_2)^{\otimes 2}_{h\Sigma_2} \xrightarrow{x^{\otimes 2}_{h\Sigma_2}} HF_2 \otimes L^{\otimes 2}_{h\Sigma_2}$$

(1): $x \in H_j(L) \longleftrightarrow \Sigma^j HF_2 \xrightarrow{x} HF_2 \otimes L$

Both $\bar{Q}^j x$ & $[x, x]$ are induced by

$$\Sigma^k HF_2 \otimes \Sigma^k HF_2 \longrightarrow (\Sigma^k HF_2)^{\otimes 2}_{h\Sigma_2} \xrightarrow{x^{\otimes 2}_{h\Sigma_2}} HF_2 \otimes L^{\otimes 2}_{h\Sigma_2} \xrightarrow{\Sigma \text{ str map}} HF_2 \otimes L$$

$\Rightarrow$ by inspection ✓

(2): $x \in H_i(L)$, $y \in H_j(L)$. WTS $[x, \bar{Q}^k y] = 0$

- $k < j$ ✓
- $k = j$: $[x, \bar{Q}^k y] = [x, [y, y]] = 0$ by Jacobi + (graded) comm. mod 2 ✓
- $k > j$:

Let $q_{k-j} : \Sigma^{k-j} HF_2 \longrightarrow HF_2 \wedge \Sigma^\infty_+ B\Sigma_2$

$H_* q_{k-j} : 1 \longmapsto e_{k-j}$ (pick out $(k-j)^{th}$ cell)

Let $[i] := \Sigma^i HF_2$, $\bar{L} = HF_2 \otimes L$, $\overline{B\Sigma_2} = HF_2 \wedge \Sigma^\infty_+ B\Sigma_2$

$$\begin{tikzcd}
{[i+j+k-2]} \arrow[d, "id_{[i-1]} \otimes \Sigma^{-1} q_{k-j} \otimes id_{[2j]}"] & \\
\Sigma^{-1}[i] \otimes \Sigma^{-1}(\overline{B\Sigma_2} \otimes [2j]) \arrow[d, "\sim"', "(\forall G \curvearrowright X \text{ inv} \Rightarrow X_{hG} \simeq X \otimes BG_+) \ \& \ \Sigma_2 \curvearrowright S^{-1} \text{ inv}"] & \\
\partial_2 \otimes (\partial_1 \otimes [i]) \otimes (\partial_2 \otimes [j])^{\otimes 2}_{h\Sigma_2} \arrow[r, "\text{str map}"] \arrow[d, "x, y"] \arrow[dr, phantom, "\circ"] & (\partial_3 \otimes [i+2j])_{h(\Sigma_1 \times \Sigma_2)} \arrow[d, "x, y"] \\
\partial_2 \otimes (\partial_1 \otimes \bar{L}) \otimes (\partial_2 \otimes \bar{L})^{\otimes 2}_{h\Sigma_2} \arrow[r] \arrow[d] \arrow[dr, phantom, "\circ"] & (\partial_3 \otimes \bar{L})_{h(\Sigma_1 \times \Sigma_2)} \arrow[d] \\
\partial_2 \otimes \bar{L} \otimes \bar{L} \arrow[r] & \bar{L}
\end{tikzcd}$$

$$= [x, \bar{Q}^k y] = \underbrace{[i+j+k-2] \longrightarrow \left(\partial_3 \wedge [i+2j]\right)_{h(\Sigma_1 \times \Sigma_2)} \longrightarrow \Sigma}_{\text{want this}^{?} \text{ nullhom}}$$

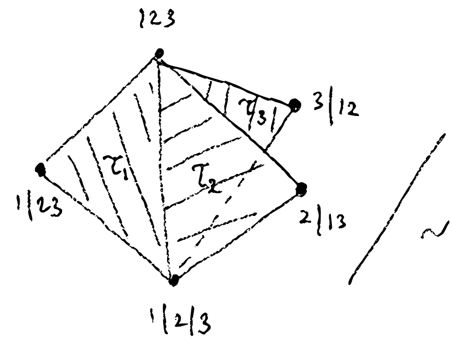
ETS $\left(\partial_3 \wedge [i+2j]\right)_{h(\Sigma_1\times\Sigma_2)}$ concentrated in deg $i+2j-2 < i+j+k-2$.

$\Sigma_1 \times \Sigma_2 \curvearrowright \partial_3$ freely $\Rightarrow \left(\partial_3 \wedge [i+2j]\right)_{h(\Sigma_1\times\Sigma_2)} = H\mathbb{F}_2 \wedge \Sigma^{i+2j}(\partial_3)_{h(\Sigma_1\times\Sigma_2)}$

Want ∂_3 as $(\Sigma_1\times\Sigma_2)$-spectrum:

$\partial_3 = D\Sigma^\infty P_3$

$\Sigma_1 \times \Sigma_2 \curvearrowright P_3$ fix τ_1, swap τ_2 & τ_3



$\Rightarrow P_3 \underset{\Sigma_1\times\Sigma_2}{\simeq} \Sigma^2 \underbrace{\Sigma^\infty_+ \Sigma_2}_{\text{regular rep of } \Sigma_2}$
(equivariantly)

$\Rightarrow \partial_3 \underset{\Sigma_1\times\Sigma_2}{\simeq} \Sigma^{-2}\Sigma^\infty_+\Sigma_2$

$\Rightarrow (\partial_3)_{h(\Sigma_1\times\Sigma_2)} \simeq S^{-2}$

$\Rightarrow \left(\partial_3 \wedge [i+2j]\right)_{h(\Sigma_1\times\Sigma_2)} \simeq H\mathbb{F}_2 \wedge S^{i+2j-2}$

$\square$

5 Homology of free sLie-algs on s.c. spaces (Camarena Section 7)

Main Thm: X s.c. space. Then

$$H_*(\mathrm{Free}_{s\mathcal{L}}(\Sigma^\infty X)) \xleftarrow{\simeq} \underbrace{s\mathcal{L}_{\bar{R}}}_{\text{free allowable } \bar{R}\text{-sLie-alg}}(\tilde{H}_* X)$$

free allowable $\bar{R}$ - sLie - alg
$\parallel$
allowable $\bar{R}$-mod + sLie-alg str s.t. (1) $\bar{Q}^{|x|}(x) = [x, x]$
(2) $[x, \bar{Q}^k y] = 0$

Note: need s.c. to use Hilton Milnor Thm

Case 1: $X = S^{n \geq 2}$

$\mathbb{D}_k(S^n)$ only non-0 for k = power of 2

Behrens 1.5.1 $\Rightarrow$ $H_*(\mathrm{Free}_{s\mathcal{L}}(S^n)) = \bigoplus_{k \geq 0} H_*(\mathbb{D}_{2^k}(S^n)) = \bar{R}_n\{\iota_n\} = s\mathcal{L}_{\bar{R}}(\tilde{H}_*(S^n))$ ✓

$H_n(\mathbb{D}_1(S^n)) \subseteq$

Actually $x_n \in H_n(\mathbb{D}_1(S^n))$, $x_n \longleftarrow \iota_n$

Case 2: $X = S^{d_1} \vee S^{d_2} \vee \dots \vee S^{d_k}$, $d_i \geq 2 \ \forall i$

Hilton-Milnor (David's talk) $\Rightarrow$ $\mathrm{Free}_{s\mathcal{L}}(S^{d_1} \vee \dots \vee S^{d_k}) = \bigvee_{w \in B_k} \mathrm{Free}_{s\mathcal{L}}\left(S^{1 + \sum_i m_i(w)(d_i - 1)}\right)$

B_k: basic prods in k letters

$\Rightarrow$ reduce to case 1

Case 3: X any s.c space : follows formally from case 2

- If X arbitrary wedge of spheres,
  - $=$ filtered colim of finite wedge of spheres (case 2)
  - H_* & Free_{sL} comm. w/ filt. colim

  $\Rightarrow$ done by case 2

- If X any s.c.,

  Pick basis $\{x_j\}$ of $\tilde{H}_*(X)$

$$\bigvee_j S^{|x_j|} \xrightarrow{\oplus x_j} HF_2 \wedge \Sigma^\infty X \rightsquigarrow \bigvee_j \Sigma^{|x_j|} HF_2 \xrightarrow[\text{in } \mathrm{Mod}_{HF_2}]{\sim} HF_2 \wedge \Sigma^\infty X$$

$\exists$ nat trans $\psi : sL_{\bar{R}}(\pi_*(-)) \longrightarrow \pi_*\left(\mathrm{Free}_{HF_2 \wedge \partial_*(id)}(-)\right)$

of functors $\mathrm{Mod}_{HF_2} \longrightarrow$ allowable $\bar{R}$-s.Lie-algs

induced by univ prop of $sL_{\bar{R}}$ = free allowable $\bar{R}$-s.Lie-algs

$$\begin{tikzcd}
sL_{\bar{R}}\left(\pi_*\left(\bigvee_j \Sigma^{|x_j|} HF_2\right)\right) \arrow[r, "\psi_{\bigvee_j \Sigma^{|x_j|} HF_2}", "\sim \text{ by case 2}"'] \arrow[d, "\wr"'] \arrow[dr, phantom, "\circ"] & \pi_*\left(\mathrm{Free}_{HF_2 \wedge \partial_*(id)}\left(\bigvee_j \Sigma^{|x_j|} HF_2\right)\right) \arrow[d, "\wr"] \\
sL_{\bar{R}}\left(\pi_*\left(HF_2 \wedge \Sigma^\infty X\right)\right) \arrow[r, "\Rightarrow \sim", "\psi_{HF_2 \wedge \Sigma^\infty X}"'] & \pi_*\left(\mathrm{Free}_{HF_2 \wedge \partial_*(id)}\left(HF_2 \wedge \Sigma^\infty X\right)\right)
\end{tikzcd}$$

$\square$