DIFFERENTIATING THE EHP SEQUENCE AND THE HILTON-MILNOR THEOREM

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These are the notes for my talk at MIT juvitop.

1. Motivation

Recall that the spectral Lie operad is defined by the symmetric sequence $(\partial_0, \partial_1, \dots)$, so that

$$\mathrm{Free}_{\mathrm{Lie}}(Y) = \bigoplus_{n=0}^{\infty} (\partial_n \otimes Y^{\otimes n})_{h\Sigma_n}.$$

We wish to understand the free spectral Lie algebra on spheres

$$\mathrm{Free}_{\mathrm{Lie}}(S^a) \quad \text{or} \quad \mathrm{Free}_{\mathrm{Lie}}(S^{a_1} \oplus \cdots \oplus S^{a_k}).$$

Since

$$\mathrm{Free}_{\mathrm{Lie}}(\Sigma^\infty X) = \bigoplus_{n=0}^{\infty} (\partial_n \otimes \Sigma^\infty X^{\wedge n})_{h\Sigma_n} = \bigoplus_{n=0}^{\infty} \mathbb{D}_n(X),$$

we have some understanding of $\mathrm{Free}_{\mathrm{Lie}}(S^a)$ when a is odd. For example, recall the following theorem:

Theorem 1.1 ([AM99])**.** *$\mathbb{D}_n(S^{2b+1}) = 0$ unless $n = p^e$.*

We also know how to compute its $\mathbb{F}_p$-homology when n is a prime power. For even spheres, we have the following result:

Theorem 1.2. *$\mathbb{D}_n(S^{2b}) = 0$ unless $n = p^e, 2p^e$. More precisely, there is a fiber sequence*

$$\mathbb{D}_n(S^{2b-1}) \to \Sigma^{-1}\mathbb{D}_n(S^{2b}) \to \Sigma^{-1}\mathbb{D}_{n/2}(S^{4b-1})$$

where the third term should read 0 if n is odd.

For multiple generators, if $Y = Y_1 \oplus \cdots \oplus Y_k$, we have

$$(\partial_j \otimes Y^{\otimes j})_{h\Sigma_j} = \bigoplus_{n_1+\cdots+n_k=n} (\partial_j \otimes Y_1^{\otimes n_1} \otimes \cdots \otimes Y^{\otimes n_k})_{h(\Sigma_{n_1}\times\cdots\times\Sigma_{n_k})},$$

and we would like to understand the summands of RHS.

2. More Goodwillie Calculus

Lemma 2.1. *Let $F : \mathcal{S}_*^k \to \mathcal{S}_*$ be a functor, and let $\Sigma : \mathcal{S}_*^k \to \mathcal{S}_*^k$ denote the functor suspending all k inputs. If $G = \Omega F \Sigma$, then $\mathbb{D}_{(n_1,\dots,n_k)}G = \Sigma^{-1}(\mathbb{D}_{(n_1,\dots,n_k)}F)\Sigma$.*

Proof. Left exactness of $\mathbb{D}$ takes care of Ω. For Σ, it's by the construction of $P_{(n_1,\dots,n_k)}$ ([Lur17, 6.1.1.30]). $\square$

For the next lemma, let $F : \mathcal{S}_* \to \mathcal{S}_*$ and

$$G(X_1, \dots, X_k) := F(X_1 \vee \cdots \vee X_k).$$

Recall that we have

$$\mathbb{D}_k F(X) = (\partial_k F \otimes \Sigma^\infty X^{\wedge k})_{h\Sigma_k}$$

and

$$\mathbb{D}_{(1,1,\dots,1)}G(X_1, \dots, X_k) = \partial_k F \otimes \Sigma^\infty (X_1 \wedge \cdots \wedge X_k)$$

by the $\mathrm{cr}_n \leftrightarrow (-)_{h\Sigma_n}$ equivalence between symmetric multilinear functors and homogeneous functors. Similarly, we have the following lemma:

Lemma 2.2. *If F, G are as above,*

$$\mathbb{D}_{(n_1,\dots,n_k)}G(X_1,\dots,X_k) = (\partial_n F \otimes \Sigma^\infty(X_1^{\wedge n_1} \wedge \cdots \wedge X_k^{\wedge n_k}))_{h(\Sigma_{n_1}\times\cdots\Sigma_{n_k})}.$$

where $n = n_1 + \cdots + n_k$.

Next, we consider functors of the form $G(X) = F(X^{\wedge m})$.

Lemma 2.3. *If F is analytic, then $P_n G(X) = P_{\lfloor n/m \rfloor} F(X^{\wedge m})$, and*

$$\mathbb{D}_n G(X) = \mathbb{D}_{n/m} F(X^{\wedge m})$$

if m divides n and 0 if m doesn't divide n.

More generally, if $G(X_1,\dots,X_k) = F(X_1^{\wedge m_1} \wedge \cdots X_k^{\wedge m_k})$ then

$$\mathbb{D}_{(\ell m_1,\dots,\ell m_k)}G(X_1,\dots,X_k) = \mathbb{D}_\ell F(X_1^{\wedge m_1} \wedge \cdots X_k^{\wedge m_k})$$

and 0 otherwise.

Proof. We sketch the proof of the first statement following [Beh12, Lem. 2.1.3]. We shall show that

$$P_n G(X) = P_{\lfloor n/m \rfloor} F(X^{\wedge m}).$$

First, we show that the RHS is n-excisive. Writing $P_{\lfloor n/m \rfloor} F$ as a taylor tower, it is enough to show that

$$\Omega^\infty(\partial_j F \otimes (X^{\wedge m})^{\wedge j})_{h\Sigma_j}$$

is n excisive for $j \leq \lfloor n/m \rfloor$. But this is equal to

$$\Omega^\infty((\partial_j F \otimes \Sigma^\infty_+ \Sigma_{mj})_{h\Sigma_j} \otimes X^{\wedge mj})_{h\Sigma_{mj}}$$

which is mj-homogeneous. Therefore, $P_{[n/m]} F(X^{\wedge m})$ is n-excisive and we obtain a map

$$P_n G(X) \to P_{\lfloor n/m \rfloor} F(X^{\wedge m}).$$

To show that this is an equivalence, we need to do some connectivity analysis and use the following lemma. □

Lemma 2.4. *([Goo03, Prop 1.6]) Let $F, G : \mathcal{S}_* \to \mathcal{S}_*$ be functors and let $F \to G$ be a map of functors. If there exist $c, \rho \in \mathbb{Z}$ such that $F(X) \to G(X)$ is $((n+1)\operatorname{conn}(X) - c)$-connected for any $\operatorname{conn}(X) > \rho$, then $P_n F \to P_n G$ is an equivalence.*

3. Differentiating EHP Sequence

This section is essentially [AM99, Section 4.2]. Let Sq be the functor $X \mapsto X^{\wedge 2}$.

Theorem 3.1. *There are maps of functors*

$$\mathrm{id} \to \Omega\Sigma \to \Omega\Sigma\,\mathrm{Sq}$$

such that the composite is null. This is a fiber sequence when evaluated at an odd sphere.

Proof. The first map is the unit map. The second map, for connected X, comes from the James splitting

$$\Sigma\Omega\Sigma X = \bigvee_{n \geq 1} \Sigma X^{\wedge n} \to \Sigma X^{\wedge 2}.$$

See [DH21] for a detailed discussion. □

Let $F = \mathrm{fib}(\Omega\Sigma \to \Omega\Sigma\,\mathrm{Sq})$. Then, since $\mathbb{D}_n$ is left exact, we get a fiber sequence.

$$\mathbb{D}_n F(S^{2b-1}) \to \Sigma^{-1}\mathbb{D}_n(\mathrm{id})(S^{2b}) \to \Sigma^{-1}\mathbb{D}_{n/2}(S^{4b-1}).$$

Using the next lemma, we have Theorem 1.2.

Lemma 3.2. $\mathbb{D}_n(\mathrm{id})(S^{2b-1}) \to \mathbb{D}_n F(S^{2b-1})$ *is an equivalence for all n.*

Proof. We use induction on n. Consider the diagram

$$\begin{array}{ccccc} \Omega^\infty \mathbb{D}_n(\mathrm{id})(S^{2b-1}) & \longrightarrow & P_n(\mathrm{id})(S^{2b-1}) & \longrightarrow & P_{n-1}(\mathrm{id})(S^{2b-1}) \\ \downarrow & & \downarrow & & \downarrow \\ \Omega^\infty \mathbb{D}_n(F)(S^{2b-1}) & \longrightarrow & P_n(F)(S^{2b-1}) & \longrightarrow & P_{n-1}(F)(S^{2b-1}) \end{array}$$

The right vertical arrow is an equivalence by the induction hypothesis. Also, there is some $s \in \mathbb{Z}$ such that the middle vertical arrow is $((2b-1)(n+1)-s)$-connected. We can see this from the diagram

$$\begin{array}{ccc} S^{2b-1} & \longrightarrow & P_n(\mathrm{id})(S^{2b-1}) \\ \downarrow & & \downarrow \\ F(S^{2b-1}) & \longrightarrow & P_n(F)(S^{2b-1}) \end{array}$$

using that $S^{2b-1} \simeq F(S^{2b-1})$. Therefore, the left vertical arrow

$$\Omega^\infty \mathbb{D}_n(\mathrm{id})(S^{2b-1}) \to \Omega^\infty \mathbb{D}_n(F)(S^{2b-1})$$

is also $((2b-1)(n+1)-s)$-connected.

Doing the same for $S^{2b+2c-1}$ for c large, we have that

$$\Omega^\infty(\partial_n(\mathrm{id}) \otimes S^{n(2b+2c-1)})_{h\Sigma_n} \to \Omega^\infty(\partial_n(F) \otimes S^{n(2b+2c-1)})_{h\Sigma_n}$$

is $((2b+2c-1)(n+1)-s)$-connected. Since $\partial_n(\mathrm{id})$ and $\partial_n(F)$ are bounded below, if c is large enough, the map of spectra

$$(\partial_n(\mathrm{id}) \otimes S^{n(2b+2c-1)})_{h\Sigma_n} \to (\partial_n(F) \otimes S^{n(2b+2c-1)})_{h\Sigma_n}$$

is also $((2b+2c-1)(n+1)-s)$-connected. But since $(S^{2c})^n$ is a complex representation sphere of Σ_n, we have

$$\mathbb{Z} \otimes (\partial_n(\mathrm{id}) \otimes S^{n(2b+2c-1)})_{h\Sigma n} = \mathbb{Z} \otimes \Sigma^{2nc}(\partial_n(\mathrm{id}) \otimes S^{n(2b-1)})_{h\Sigma_n}$$
$$\mathbb{Z} \otimes (\partial_n(F) \otimes S^{n(2b+2c-1)})_{h\Sigma n} = \mathbb{Z} \otimes \Sigma^{2nc}(\partial_n(F) \otimes S^{n(2b-1)})_{h\Sigma_n}$$

by Thom isomorphism as $\mathbb{Z}$ is complex-oriented, so that

$$\mathbb{Z} \otimes (\partial_n(\mathrm{id}) \otimes S^{n(2b-1)})_{h\Sigma_n} \to \mathbb{Z} \otimes (\partial_n(F) \otimes S^{n(2b-1)})_{h\Sigma_n}$$

is $((2b-1)(n+1)+2c-s)$-connected. Since connectivity of bounded below spectra can be tested after applying $\mathbb{Z} \otimes -$, we have that

$$(\partial_n(\mathrm{id}) \otimes S^{n(2b-1)})_{h\Sigma_n} \to (\partial_n(F) \otimes S^{n(2b-1)})_{h\Sigma_n}$$

is $((2b-1)(n+1)+2c-s)$-connected. Taking $c \to \infty$, we have the desired result. □

4. Differentiating Hilton-Milnor

This section is from [AK98]. The following statement is the Hilton-Milnor theorem. See [DH21] for a detailed discussion.

Theorem 4.1. *There is a multiset B_k of k-tuples of nonnegative integers such that each $w = (w_1, \dots, w_k)$ appear finitely many times, and a natural equivalence*

$$\Omega\Sigma(X_1 \vee \cdots \vee X_k) = \prod_{w \in B_k} \Omega\Sigma(X_1^{\wedge w_1} \wedge \cdots \wedge X_k^{\wedge w_k})$$

for connected $X_1, \dots, X_k$.

Considering both sides as functors in $X_1, \dots, X_k$, we take $\mathbb{D}_{(n_1,\dots,n_k)}$. The LHS is

$$\Sigma^{-1}(\partial_n \otimes (\Sigma X_1)^{\wedge n_1} \wedge \cdots \wedge (\Sigma X_k)^{\wedge n_k})_{h(\Sigma_{n_1} \times \cdots \times \Sigma_{n_k})}.$$

The RHS is

$$\bigoplus_{w \in B_k, \ell w = (n_1,\dots,n_k)} \mathbb{D}_\ell(\Omega\Sigma)(X_1^{\wedge w_1} \wedge \cdots \wedge X_k^{\wedge w_k}) = \bigoplus_{w \in B_k, \ell w = (n_1,\dots,n_k)} \Sigma^{-1}(\partial_\ell \otimes (\Sigma X_1^{\wedge w_1} \wedge \cdots \wedge X_k^{\wedge w_k})^\ell)_{h\Sigma_\ell}$$

Summing up for all $(n_1, \ldots, n_k)$, we have

$$\mathrm{Free}_{\mathrm{Lie}}(\Sigma(X_1 \vee \cdots \vee X_k)) = \bigoplus_{w \in B_k} \mathrm{Free}_{\mathrm{Lie}}(\Sigma(X_1^{\wedge w_1} \wedge \cdots \wedge X_k^{\wedge w_k})).$$

References

[AK98] Greg Arone and Marja Kankaanrinta. The homology of certain subgroups of the symmetric group with coefficients in *Lie*(n), *J. Pure Appl. Algebra*, 127(1):1-14, 1998.

[AM99] Greg Arone and Mark Mahowald. The Goodwillie tower of the identity functor and the unstable periodic homotopy of spheres. *Invent. Math.*, 135(3):743-788, 1999.

[Beh12] Mark Behrens. The Goodwillie tower and the EHP sequence. *Mem. Amer. Math. Soc.*, 218(1026):xii+90, 2012.

[DH21] Sanath Devalapurkar and Peter Haine. *Doc. Math.*, 26:1423-1464, 2021.

[Goo03] Thomas G. Goodwillie. Calculus III: Taylor Series. *Geom. Topol.*, 7:645-711, 2003.

[Lur17] Jacob Lurie. *Higher algebra.* 2017.