SPECTRAL PARTITION LIE ALGEBRAS AND KOSZUL DUALITY

§0. Introduction

We've been discussing the spectral Lie operad. This is an operad in the ∞-category of spectra, which I'll denote here by $\mathbb{L}$. This in particular allows us to consider, for any $\mathbb{E}_\infty$-ring spectrum k, the ∞-category $\mathrm{Lie}_k := \mathrm{Alg}_{\mathbb{L}}(\mathrm{Mod}_k)$ of *spectral Lie algebras over k*. In this talk, we will focus on the case that k is an ordinary field.

There is a further dichotomy to this case: the characteristic of k may be zero or it may be positive. On the one hand, $\mathrm{Lie}_{\mathbb{Q}}$ has a "classical" description: it is equivalent to the ∞-category obtained by inverting quasi-isomorphisms in the category of differential graded Lie algebras (note however that this equivalence involves a shift). On the other hand, for a prime number p, the ∞-category $\mathrm{Lie}_{\mathbb{F}_p}$ is more subtle: for instance, we saw in the previous talk that there is a rich supply of operations on the homotopy groups of spectral Lie algebras over $\mathbb{F}_2$.

We've seen two characterizations of the spectral Lie operad, first as the Goodwillie derivatives of the identity functor on pointed spaces and second as the Koszul dual to the nonunital $\mathbb{E}_\infty$-operad. The latter is the one relevant to today's talk. I'll review the meaning of Koszul duality later on, but whatever it means, one might guess that accompanying this operad-level duality is a duality at the level of algebras, i.e. between spectral Lie algebras and nonunital $\mathbb{E}_\infty$-algebras. Let me begin by stating such a result that holds over $\mathbb{Q}$.

Definition 0.1. Let k be a field and let A be an augmented $\mathbb{E}_\infty$-k-algebra. Say that A is *complete local noetherian* if the following conditions hold: A is connective; $\pi_0(A)$ is a noetherian commutative ring; $\pi_0(A)$ is complete with respect to its (finitely generated) augmentation ideal, i.e. the kernel of the map $\pi_0(A) \to k$ induced by the augmentation of A; for each $i > 0$, $\pi_i(A)$ is a finitely generated $\pi_0(A)$-module. Let $\mathrm{CAlg}_k^{\mathrm{cln}}$ denote the ∞-category of complete local noetherian augmented $\mathbb{E}_\infty$-k-algebras.

Theorem 0.2. [Lurie, Pridham] *There is a fully faithful embedding $(\mathrm{CAlg}_{\mathbb{Q}}^{\mathrm{cln}})^{\mathrm{op}} \hookrightarrow \mathrm{Lie}_{\mathbb{Q}}$, with essential image given by those $L \in \mathrm{Lie}_{\mathbb{Q}}$ such that $\pi_i(L) \simeq 0$ for each $i > 0$ and $\pi_i(L)$ is finite dimensional over $\mathbb{Q}$ for each $i \leq 0$.*

Theorem 0.2 describes a partial relationship between $\mathbb{E}_\infty$-algebras and spectral Lie algebras over $\mathbb{Q}$, but this is not the whole picture. To indicate this, let me sketch another part of the picture, which comes from rational homotopy theory.

Notation 0.3. Let $\mathrm{Spc}_{\mathbb{Q}}$ denote the full subcategory of the ∞-category of spaces spanned by those spaces X that are simply connected and for which $\pi_i(X)$ is a $\mathbb{Q}$-vector space for each $i \geq 2$. Let $(\mathrm{Spc}_{\mathbb{Q}})_*$ denote the ∞-category of pointed objects in $\mathrm{Spc}_{\mathbb{Q}}$. Let $\mathrm{Spc}_{\mathbb{Q}}^{\mathrm{ft}}$ denote the full subcategory of $\mathrm{Spc}_{\mathbb{Q}}$ spanned by those objects where moreover $\pi_i(X)$ is a finite dimensional $\mathbb{Q}$-vector space for each $i \geq 2$.

Theorem 0.4. [Sullivan] *The functor $\mathrm{C}^*(-;\mathbb{Q}) : \mathrm{Spc}_{\mathbb{Q}}^{\mathrm{ft}} \to \mathrm{CAlg}_{\mathbb{Q}}^{\mathrm{op}}$ is fully faithful.*

Theorem 0.5. [Quillen] *There is a fully faithful embedding $(\mathrm{Spc}_{\mathbb{Q}})_* \hookrightarrow \mathrm{Lie}_{\mathbb{Q}}$.*

Combining Theorems 0.4 and 0.5, we see that the opposite of a certain full subcategory of *coconnective* augmented $\mathbb{E}_\infty$-$\mathbb{Q}$-algebras also embeds fully faithfully into $\mathrm{Lie}_{\mathbb{Q}}$ (the augmentation of the $\mathbb{E}_\infty$-algebra corresponds to the pointing of the space). This does not fall into the scope of Theorem 0.2, and suggests a common generalization of these results. There is indeed such a generalization (also due to Lurie and Pridham): there is an equivalence between "$\mathbb{E}_\infty$ formal moduli problems" and spectral Lie algebras over $\mathbb{Q}$; but let's not get into the precise formulation of this today.

Today's goal is to outline a proof of Theorem 0.2 and its analogue over $\mathbb{F}_p$, following the work of Brantner–Mathew. The analogue will describe $\mathrm{CAlg}_{\mathbb{F}_p}^{\mathrm{cln}}$ not in terms of spectral Lie algebras over $\mathbb{F}_p$, but a variant notion introduced by Brantner–Mathew, which they call *spectral partition Lie algebras*.

Before we dive into that, let me end the introduction with a heuristic idea behind the relationship between $\mathbb{E}_\infty$-algebras and Lie algebras.

Heuristic 0.6. Let k be a field and let A be an $\mathbb{E}_\infty$-k-algebra. Let's think geometrically: there are some geometric objects $\mathrm{Spec}(k)$ and $\mathrm{Spec}(A)$ associated to k and A, and the former can be thought of as a point. An augmentation $A \to k$ corresponds to a map $\mathrm{Spec}(k) \to \mathrm{Spec}(A)$, i.e. a point of $X := \mathrm{Spec}(A)$. Given such a basepoint, we can pass to the loop space $\Omega X = \mathrm{Spec}(k \otimes_A k)$. There is a group structure on the loop space, so we may think of it as analogous to a Lie group. This analogy suggests that the tangent space of ΩX at its identity should be some sort of a Lie algebra.

The idea is that, if A is sufficiently small, i.e. $\mathrm{Spec}(A)$ sufficiently localized around its basepoint, then this construction should not lose information: that is, we should be able to recover A from the Lie algebra we end up with. This is the upshot of the main results we'll discuss today, but we won't phrase it in these geometric terms. Rather, we'll just extract the desired Lie algebra directly from A, as the dual of its "cotangent fiber" (note that, since we won't pass to the loop space, the relevant notion of Lie algebra is off by a shift from the classical notion; this shift is the same as the one mentioned above in the comparison of spectral Lie algebras with differential graded Lie algebras over $\mathbb{Q}$).

§1. A Koszul duality warm-up

Before we discuss Koszul duality between operads and algebras over operads, let's warm up in a simpler setting. Let k be a field, let R be an associative k-algebra spectrum, and let $\epsilon : R \to k$ be an augmentation. Then we have an adjunction

$$F : \mathrm{Mod}_R \rightleftarrows \mathrm{Mod}_k : G$$

given by base change and restriction along ϵ. We can ask ourselves:

Question 1.1. Can we recover an R-module N from the k-module $F(N) = k \otimes_R N$? Or rather, is there some natural additional structure on the latter that can be used to recover the former?

Abstract nonsense tells us one natural structure to consider: the composition FG is a comonad on Mod_k, and for an R-module N, the k-module $F(N)$ is naturally a comodule over this comonad. In the current setting, all this can be rephrased in the following way:

- For $M \in \mathrm{Mod}_k$, we have $FG(M) \simeq k \otimes_R M \simeq (k \otimes_R k) \otimes_k M$. The comonad structure on FG comes from the natural coalgebra structure on $\mathrm{Bar}(R) := k \otimes_R k$; this coalgebra structure was explained in a previous talk (using the notion of "coendomorphism object"), but for example, the comultiplication map is the composition

$$k \otimes_R k \simeq k \otimes_R R \otimes_R k \xrightarrow{\epsilon} k \otimes_R k \otimes_R k.$$

- For $N \in \mathrm{Mod}_R$, the k-module $F(N) = k \otimes_R N$ is naturally a comodule over $\mathrm{Bar}(R) = k \otimes_R k$; for example, the coaction map is the composition

$$k \otimes_R N \simeq k \otimes_R R \otimes_R N \xrightarrow{\epsilon} k \otimes_R k \otimes_R N \simeq (k \otimes_R k) \otimes_k (k \otimes_R N).$$

To summarize, the adjunction $F \dashv G$ canonically factors through another adjunction

$$F' : \mathrm{Mod}_R \rightleftarrows \mathrm{cMod}_{\mathrm{Bar}(R)} : G'.$$

To address Question 1.1, we can ask how close this new adjunction is to being an equivalence. This can be tested using the following "comonadicity" result:

Theorem 1.2. [Beck, Lurie] *If $F : \mathrm{Mod}_R \to \mathrm{Mod}_k$ preserves the limits (i.e. totalizations) of a certain class of cosimplicial diagrams*[1]*, then $G' : \mathrm{cMod}_{\mathrm{Bar}(R)} \to \mathrm{Mod}_R$ is fully faithful. If F is also conservative, then G' (and hence F') is an equivalence.*

Example 1.3. Take $R = k[t]$, with the augmentation $R \to k$ sending $t \mapsto 0$. Then there is a cofiber sequence of R-modules $R \to R \to k$, where the first map is multiplication by t. From this we deduce two

[1]Namely, the F-split cosimplicial diagrams; but we won't get into the details of this today.

things:

- There is a cofiber sequence of k-modules $k \to k \to k \otimes_R k$ where the first map is zero. Thus, $\mathrm{Bar}(R) = k \otimes_R k \simeq k \oplus k[1]$.
- The base change functor $F : \mathrm{Mod}_R \to \mathrm{Mod}_k$ preserves *all* limits, as k is a perfect R-module.

Using Theorem 1.2, the second of these implies that in the adjunction $F' : \mathrm{Mod}_R \rightleftarrows \mathrm{cMod}_{\mathrm{Bar}(R)} : G'$, the right adjoint G' is fully faithful. And the first suggests how to think about comodules over $\mathrm{Bar}(R)$: they are in particular k-modules M equipped with a map $d : M \to M[1]$; more careful thought shows that d is furthermore a "differential" (it is square-zero, in a homotopy-coherent sense).

So, in this example, the discussion above is telling us that to attempt to recover a $k[t]$-module from its quotient mod t, we should remember a differential on the latter, and the recovery will work up to a certian (Bousfield) localization of the ∞-category of $k[t]$-modules (arising because F is not conservative). This should sound sensical: the differential is the "t-Bockstein", and the relevant localization is given by t-completion. (One can also analyze base change along the map $\mathbb{Z}_p \to \mathbb{F}_p$ in a similar fashion, and see the usual Bockstein appear, but this doesn't fit exactly into the discussion of this section, since $\mathbb{Z}_p$ is not an $\mathbb{F}_p$-algebra.)

§2. The Koszul duality we wanted

We will now repeat the discussion of the previous section but with algebras over k replaced by operads over k and modules over algebras replaced by algebras over operads. Let's first recall the setup for operads.

Notation 2.1. Let k be a field still, let $\mathbb{F}$ denote the groupoid of finite sets, and let SSeq_k denote the ∞-category $\mathrm{Fun}(\mathbb{F}, \mathrm{Mod}_k)$ of *symmetric sequences of k-modules*. For $X \in \mathrm{SSeq}_k$ and $n \geq 0$, we write $X(n) := X(\{1, \ldots, n\})$ (where this means $X(\varnothing)$ for $n = 0$), which is a k-module with Σ_n-action.

There is a fully faithful embedding $\iota : \mathrm{Mod}_k \hookrightarrow \mathrm{SSeq}_k$ with essential image those symmetric sequences X such that $X(n) \simeq 0$ for $n > 0$. We may abuse notation and just regard Mod_k as a full subcategory of SSeq_k via this embedding.

There is a Day convolution product $\otimes$ on SSeq_k, given by

$$(X \otimes Y)(T) \simeq \bigoplus_{T = T_0 \sqcup T_1} X(T_0) \otimes_k Y(T_1)$$

for $X, Y \in \mathrm{SSeq}_k$ and $T \in \mathbb{F}$. This extends to a symmetric monoidal structure, which restricts to the usual symmetric monoidal structure on Mod_k via the embedding ι.

There is also a composition product $\circ$ on SSeq_k, given by

$$X \circ Y \simeq \bigoplus_{n \geq 0} (X(n) \otimes Y^{\otimes n})_{\mathrm{h}\Sigma_n}$$

for $X, Y \in \mathrm{SSeq}_k$; in this formula, $\otimes$ refers to the Day convolution product of symmetric sequences, and the k-module $X(n)$ is regarded as a symmetric sequence via the embedding ι. The composition product extends to a (non-symmetric) monoidal structure on SSeq_k, where the unit object is the symmetric sequence $\mathbb{O}_{\mathrm{triv}}$ which has $\mathbb{O}_{\mathrm{triv}}(1) \simeq k$ and $\mathbb{O}_{\mathrm{triv}}(n) \simeq 0$ for $n \geq 1$. An algebra (resp. coalgebra) in SSeq_k with respect to the composition product is called an *operad over k* (resp. *cooperad over k*).

Note in the above formula for the composition product that if Y is in the essential image of ι, then so too is $X \circ Y$. Thus the composition product also determines an action of SSeq_k on Mod_k (and the above formula for the composition product shows that this action is given by the expected construction).

For $\mathbb{O}$ an operad over k, an $\mathbb{O}$-*algebra* is a module over $\mathbb{O}$ in Mod_k, i.e. a k-module A together with a map

$$\mathbb{O} \circ A = \bigoplus_{n \geq 0} (\mathbb{O}(n) \otimes A^{\otimes n})_{\mathrm{h}\Sigma_n} \to A$$

and additional coherence data. We let $\mathrm{Alg}_{\mathbb{O}}$ denote the ∞-category of $\mathbb{O}$-algebras.

For $\mathbb{O}$ a cooperad over k, a *divided power* $\mathbb{O}$-*coalgebra* is a comodule over $\mathbb{O}$ is Mod_k, i.e. a k-module

A together with a map

$$A \to \mathbb{O} \circ A = \bigoplus_{n \geq 0} (\mathbb{O}(n) \otimes A^{\otimes n})_{h\Sigma_n}$$

and additional coherence data. We let $\mathrm{cAlg}^{\mathrm{pd}}_{\mathbb{O}}$ denote the ∞-category of divided power $\mathbb{O}$-coalgebras.

Now, let $\mathbb{O}$ be an operad over k and let $\epsilon : \mathbb{O} \to \mathbb{O}_{\mathrm{triv}}$ be an augmentation, and let's replicate the discussion from §1 in this setting. We have an adjunction

$$F : \mathrm{Alg}_{\mathbb{O}} \rightleftarrows \mathrm{Alg}_{\mathbb{O}_{\mathrm{triv}}} \simeq \mathrm{Mod}_k : G,$$

where G is given by restriction along ϵ and F is given by base change. Let me spell out what base change means: for $A \in \mathrm{Alg}_{\mathbb{O}}$, we have

$$F(A) = \mathbb{O}_{\mathrm{triv}} \circ_{\mathbb{O}} A \simeq \operatorname*{colim}_{[n] \in \Delta} \mathbb{O}_{\mathrm{triv}} \circ \mathbb{O}^{\circ n} \circ A \simeq \operatorname*{colim}_{[n] \in \Delta} \mathrm{Free}^{(n)}_{\mathbb{O}}(A),$$

where $\mathrm{Free}_{\mathbb{O}} = \mathbb{O} \circ -$ is the free $\mathbb{O}$-algebra functor and $\mathrm{Free}^{(n)}_{\mathbb{O}}$ denotes its n-th iterate. The bar contruction $\mathrm{Bar}(\mathbb{O}) = \mathbb{O}_{\mathrm{triv}} \circ_{\mathbb{O}} \mathbb{O}_{\mathrm{triv}}$ has a canonical cooperad structure, and the adjunction $F \dashv G$ factors through another,

$$F' : \mathrm{Alg}_{\mathbb{O}} \rightleftarrows \mathrm{cAlg}^{\mathrm{pd}}_{\mathrm{Bar}(\mathbb{O})} : G'.$$

Let's now specialize to the case of interest: fix $\mathbb{O}$ to be the nonunital $\mathbb{E}_\infty$-operad, which has $\mathbb{O}(0) \simeq 0$ and $\mathbb{O}(n) \simeq k$ (with trivial Σ_n-action) for $n \geq 0$, and has a unique augmentation. Algebras over $\mathbb{O}$ are nonunital $\mathbb{E}_\infty$-k-algebras, which we may identify with augmented $\mathbb{E}_\infty$-k-algebras (by tacking on a unit or lopping it off). Denote $\mathrm{Bar}(\mathbb{O})$ by $\mathbb{L}^\vee$: this is the cooperad dual to the spectral Lie operad $\mathbb{L}$; from earlier talks, we know that $\mathbb{L}^\vee(n) \simeq \mathbb{L}(n)^\vee$ is given by $k \otimes \Sigma^{\infty+1} \Pi_n^\diamond$, where $\Pi_n^\diamond$ is the unreduced suspension of the n-th partition complex. For this case, let us replace the symbol G by sqz, for "square-zero multiplication", and the symbol F by cot, for "cotangent fiber". So we have adjunctions

$$\mathrm{cot} : \mathrm{CAlg}^{\mathrm{aug}}_k \rightleftarrows \mathrm{Mod}_k : \mathrm{sqz}, \qquad \mathrm{cot}' : \mathrm{CAlg}^{\mathrm{aug}}_k \rightleftarrows \mathrm{cAlg}^{\mathrm{pd}}_{\mathbb{L}^\vee} : \mathrm{sqz}'.$$

We can now state the first main result.

Notation 2.2. Let $\mathrm{Mod}^{\mathrm{ft}}_{k,\geq 0}$ denote the full subcategory of Mod_k spanned by those k-modules M that are connective and such that $\pi_i(M)$ is finite dimensional over k for each $i \geq 0$.

Theorem 2.3. [Brantner–Mathew] *Firstly:*

- *The adjunction* $\mathrm{cot} : \mathrm{CAlg}^{\mathrm{aug}}_k \rightleftarrows \mathrm{Mod}_k : \mathrm{sqz}$ *restricts to an adjunction* $\mathrm{cot} : \mathrm{CAlg}^{\mathrm{cln}}_k \rightleftarrows \mathrm{Mod}^{\mathrm{ft}}_{k,\geq 0} :$ sqz.

It follows that the action of $\mathbb{L}^\vee$ *on* Mod_k *restricts to one on* $\mathrm{Mod}^{\mathrm{ft}}_{k,\geq 0}$*, so that it makes sense to form the* ∞*-category* $\mathrm{cAlg}^{\mathrm{pd}}_{\mathbb{L}^\vee}(\mathrm{Mod}^{\mathrm{ft}}_{k,\geq 0})$ *of divided power* $\mathbb{L}^\vee$*-coalgebras in* $\mathrm{Mod}^{\mathrm{ft}}_{k,\geq 0}$*. With this in mind, secondly:*

- *The induced adjunction* $\mathrm{cot}' : \mathrm{CAlg}^{\mathrm{cln}}_k \rightleftarrows \mathrm{cAlg}^{\mathrm{pd}}_{\mathbb{L}^\vee}(\mathrm{Mod}^{\mathrm{ft}}_{k,\geq 0}) : \mathrm{sqz}'$ *is an equivalence.*

Unfortunately, I won't say anything about the proof of Theorem 2.3 today, except that it goes by verifying the criteria of Theorem 1.2. In any case, this result tells us that we can recover a complete local noetherian augmented $\mathbb{E}_\infty$-k-algebra from its cotangent fiber equipped with a divided power $\mathbb{L}^\vee$-coalgebra structure. Let us now translate this into something closer to a Lie algebra structure, by dualizing.

Notation 2.4. Let $\mathrm{Mod}^{\mathrm{ft}}_{k,\leq 0}$ denote the full subcategory of Mod_k spanned by those k-modules M that are coconnective and such that $\pi_i(M)$ is finite dimensional over k for each $i \leq 0$.

Note that linear duality supplies an equivalence $(-)^\vee : \mathrm{Mod}^{\mathrm{ft}}_{k,\geq 0} \simeq (\mathrm{Mod}^{\mathrm{ft}}_{k,\leq 0})^{\mathrm{op}}$. What does a divided power $\mathbb{L}^\vee$-coalgebra structure transfer to along this equivalence? The answer is something like "a divided power algebra over the dual operad $\mathbb{L}$" (which is not the same, in general, as a usual algebra over $\mathbb{L}$, i.e. a spectral Lie algebra). To formulate this precisely, we can work in the language of monads and comonads. A divided power $\mathbb{L}^\vee$-coalgebra structure is a coalgebra structure for the comonad

$$M \mapsto \mathrm{cot}(\mathrm{sqz}(M)) \simeq \mathbb{L}^\vee \circ M \simeq \bigoplus_{n \geq 0} (\mathbb{L}^\vee(n) \otimes M^{\otimes n})_{h\Sigma_n}.$$

This comonad on $\mathrm{Mod}^{\mathrm{ft}}_{k,\geq 0}$ dualizes to a monad T on $\mathrm{Mod}^{\mathrm{ft}}_{k,\leq 0}$, given by

$$M \mapsto \cot(\mathrm{sqz}(M^\vee))^\vee \simeq \prod_{n\geq 0}(\mathbb{L}(n)\otimes M^{\otimes n})^{\mathrm{h}\Sigma_n} \simeq \bigoplus_{n\geq 0}(\mathbb{L}(n)\otimes M^{\otimes n})^{\mathrm{h}\Sigma_n}$$

(the last equivalence follows from some finiteness considerations). It follows that linear duality determines an equivalence $\mathrm{cAlg}^{\mathrm{pd}}_{\mathbb{L}^\vee}(\mathrm{Mod}^{\mathrm{ft}}_{k,\geq 0}) \simeq \mathrm{Alg}_T(\mathrm{Mod}^{\mathrm{ft}}_{k,\leq 0})^{\mathrm{op}}$, and we can rephrase the result above as follows.

Corollary 2.5. *There is an equivalence of ∞-categories*

$$\cot^\vee : \mathrm{CAlg}^{\mathrm{cln}}_k \simeq \mathrm{Alg}_T(\mathrm{Mod}^{\mathrm{ft}}_{k,\leq 0})^{\mathrm{op}}.$$

This is the result that was promised in the introduction, describing complete local noetherian augmented $\mathbb{E}_\infty$-k-algebras in terms of some kind of Lie algebra structure over k (for any field k). The "kind of Lie algebra structure" is encoded by the monad T above, and is a "divided power version" of a spectral Lie algebra structure (in the sense that we've replaced Σ_n-orbits with Σ_n-fixed points in the definition).

Remark 2.6. In some situations, Σ_n-orbits and Σ_n-fixed points canonically agree, and then the above divide power variant of spectral Lie algebras agrees with usual spectral Lie algebras. This happens in particular when k has characteristic zero. In this way, Corollary 2.5 recovers Theorem 0.2. It also happens in the "intermediate characteristics" of chromatic homotopy theory, which is the subject of the next talk.

Let me finish by tying up one loose end: so far, the monad T is only defined on the subcategory $\mathrm{Mod}^{\mathrm{ft}}_{k,\leq 0}$ of Mod_k. This issue is addressed by the following further result:

Theorem 2.7. [Brantner–Mathew] *The monad T extends canonically to a sifted colimit–preserving monad $\mathbb{L}^\pi_k$ on Mod_k. For $M \in \mathrm{Mod}_k$ bounded above, there is a natural equivalence*

$$\mathbb{L}^\pi_k(M) \simeq \bigoplus_{n\geq 0}(\mathbb{L}(n)\otimes M^{\otimes n})^{\mathrm{h}\Sigma_n} \simeq \bigoplus_{n\geq 0}(k^{\Sigma^{\infty+1}\Pi^\diamond_n}\otimes M^{\otimes n})^{\mathrm{h}\Sigma_n}.$$

Again, I unfortunately will say nothing substantial about the proof, but the idea is that there are testable conditions (the preservation of certain colimits) for a functor on $\mathrm{Mod}^{\mathrm{ft}}_{k,\leq 0}$ to extend to a sifted colimit–preserving functor on Mod_k.

Definition 2.8. Algebras in Mod_k for the monad $\mathbb{L}^\pi_k$ of Theorem 2.7 are called *spectral partition Lie algebras over k*. Let Lie^π_k denote the ∞-category $\mathrm{Alg}_{\mathbb{L}^\pi_k}(\mathrm{Mod}_k)$ of spectral partition Lie algebras over k.

To sum up, Corollary 2.5 expresses a partial Koszul duality between $\mathbb{E}_\infty$-algebras and spectral partition Lie algebras over any field k. More generally, for any augmented $\mathbb{E}_\infty$-k-algebra A, $\cot(A)^\vee$ is naturally a spectral partition Lie algebra. However, the resulting functor $\cot^\vee : \mathrm{CAlg}^{\mathrm{aug}}_k \to (\mathrm{Lie}^\pi_k)^{\mathrm{op}}$ is not an equivalence: the finiteness conditions in Corollary 2.5 are necessary. Nevertheless, as alluded to in §0, there is a generalization of Corollary 2.5: namely, there is an equivalence between "$\mathbb{E}_\infty$ formal moduli problems" and spectral partition Lie algebras over any field k.