Juvitop Sp2022

Overview

§0. Motivation

Classical operadic Koszul duality.

[Quillen; Moore] Over $\mathbb{Q}$

$$\left\{\begin{matrix}\text{1-connected}\\ \text{cocommutative coalg.}\end{matrix}\right\} \underset{\text{Bar}}{\overset{\text{Cobar}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \left\{\begin{matrix}\text{connected}\\ \text{dg Lie alg}\end{matrix}\right\}$$

$\rightsquigarrow$ equivalence of homotopy categories.

[Ginzburg-Kapranov] This reflects the Koszul duality between the commutative operad Comm and the Lie operad Lie (via a bar construction).

Q: Can we do this in Spectra?

[Lurie]: Any operad $\mathcal{O}$ in Spectra has a Koszul dual operad $\mathbb{D}\,\mathrm{Bar}(1, \mathcal{O}, 1)$.

$$\mathrm{Comm} \rightsquigarrow \mathbb{E}_\infty^{(nu)} = (0, \mathbb{S}, \mathbb{S}, \cdots)$$

$$\mathrm{Lie} \rightsquigarrow \underbrace{\mathbb{D}\,\mathrm{Bar}(1, \mathbb{E}_\infty, 1)}_{\text{The spectral Lie operad}} \simeq (0, \underbrace{\partial_2(\mathrm{id})}_{\text{What is this?}}, \partial_3(\mathrm{id}), \cdots)$$

§1. Goodwillie Calculus

$F : Top_* \to Top_*$ a nice functor.

Goodwillie constructed a tower of fibrations

$$
\begin{tikzcd}
 & & & & \Omega^\infty\Sigma^\infty F \arrow[dl, "\simeq"'] & \\
\cdots \arrow[r] & P_n F \arrow[r] & P_{n-1}F \arrow[r] & \cdots \arrow[r] & P_1 F \arrow[r] & P_0 F \simeq * \\
\mathrm{hofib} = & D_n F \arrow[u] & D_{n-1}F \arrow[u] & & D_1 F \arrow[u] &
\end{tikzcd}
$$

$\leftarrow$ Goodwillie layers

whose inverse limit is F in good cases.

In analogy to the Taylor expansion $f(x) = \sum \frac{f'(x)\,x^n}{n!}$,

$$D_n F(X) \simeq \Omega^\infty(\underbrace{\partial_n F}_{\uparrow} \otimes X^{\otimes n})_{h\Sigma_n}$$

"$f^{(n)}(x)$", nth Goodwillie derivative, a naive Σ_n-spectrum.

<u>e.g.</u> $F = id : Top_* \to Top_*$.

[Johnson, Arone-Mahowald] nth partition complex

$$D_n(id)(X) = \Omega^\infty(\mathbb{D}(\Pi_n) \otimes \Sigma^\infty X^{\otimes n})_{h\Sigma_n}$$

Spanier-Whitehead dual.

So $\partial_n(id) \simeq \mathbb{D}(\Pi_n)$

§2. Classical Computations

[AM]: $X = S^i$, i odd.

Then $D_n(\mathrm{id})(S^i) \simeq *$ if n is not a power of some prime p

If $n = p^k$, $k > 0$, then $D_{p^k}(\mathrm{id})(S^i)$ is p-primary torsion.

They also computed the mod p homology, $\forall p$.

Rmk. To get similar results for even sphere, differentiate the EHP sequence.

[Arone-Dwyer, Mitchell-Priddy, Kuhn, etc.]

$D_{p^k}(S^i) \simeq \Omega^\infty(\Sigma^{k-i} L(k)_i)$ is the stable summand of the Thom complex $(B(\mathbb{Z}/p)^k)^{i\bar{\rho}_k}$ reduced real regular rep. associated to the Steinberg idempotent in the group ring of $GL_k(\mathbb{Z}/p)$.

§3. Back to the <u>Spectral Lie Operad</u>!

[Ching, Salvatore]

$\partial_*(\mathrm{id}) \simeq \mathbb{D}\,\mathrm{Bar}(1, \Phi, 1)$ as sym. sequences.

[Ching] constructed explicit cooperad structure on

$\mathrm{Bar}(1, \Phi, 1)$, so $\partial_*(\mathrm{id})$ is an operad.

Later on, more conceptual ways to see this:

[Arone-Ching, Lurie]

Chain rule: $\partial_*(F \circ G) \simeq \partial_* F \underset{\partial_*(\mathrm{id})}{\circ} \partial_* G$.

[Lurie] Operadic Koszul duality.

<u>Q</u>. Where is the "Lie"?

L: algebra over $S\mathcal{L} := \partial_*(\mathrm{id})$.

Then $H_*(L)$ has a shifted Lie bracket

$$[\,,\,] : H_m(L) \otimes H_n(L) \to H_{m+n-1}(L)$$

[Antolín-Camarena, Brantner]:

At the level of spectra, the shifted Lie bracket is already present:

$$\begin{tikzcd}
\mathbb{S}^{-1} \otimes L^{\otimes 2} \arrow[r] & \mathbb{S}^{-1} \otimes (L^{\otimes 2})_{h\Sigma_2} \arrow[r, "\simeq"] & \partial_2(\mathrm{id}) \otimes (L^{\otimes 2})_{h\Sigma_2} \arrow[d, "\wr"'] \\
 & & (\partial_2(\mathrm{id}) \otimes L^{\otimes 2})_{h\Sigma_2} \arrow[d, "\text{weight 2 structural map } \xi_2"'] \\
 & & L
\end{tikzcd}$$

Here
- $\partial_2(\mathrm{id}) \simeq \mathbb{S}^{-1}$ with trivial Σ_2-action
- $\mathrm{Free}^{\mathcal{SL}}(L) = \bigoplus \partial_n(\mathrm{id}) \otimes_{h\Sigma_n} L^{\otimes n} \longrightarrow L$ is the structural map.

- The Jacobi identity can also be checked at the level of spectra.

§4. Other structures on $H_*(L; \mathbb{F}_2)$?

Motivation: For A an $\mathbb{E}_\infty$-alg., $H_*(A)$ has a product and Dyer-Lashof operations Q^i.

[Behrens] Constructed Dyer-Lashof-like unary operations $\bar{Q}^i$ on $H_*(L; \mathbb{F}_2)$ for $L \in \mathrm{Alg}_{s\mathcal{L}}$ and computed relations.

$\rightsquigarrow$ algebra of operations $\bar{\mathcal{R}}$

- [AM] can be expressed as

$$H_*(D_{2^k}(S^n)) = \mathrm{Free}^{\bar{\mathcal{R}}}(\tilde{H}_*(S^n))$$

[Antolín-Camarena] Checked compatibility of brackets with $\bar{Q}^i$'s to get the target catgory of $H_*(L)$ and computed the homology of $\mathrm{Free}^{s\mathcal{L}}(X)$

(Technique: Differentiate Hilton-Milnor to reduce X a wedge of spheres to a single sphere.)

§J. Q: Are spectral Lie algebras Koszul dual to $\mathbb{E}_\infty$-algebras?

No! Koszul duality for algebras over operads are subtle: For $\mathcal{O}$ any operad

$$\mathcal{O} \overset{\mathrm{Bar}}{\longmapsto} \underset{\text{cooperad}}{\mathrm{Bar}(1,\mathcal{O},1)} \xrightarrow{\mathbb{D}} \mathbb{D}\mathrm{Bar}(1,\mathcal{O},1)$$

$$\mathrm{Alg}_{\mathcal{O}} \xrightarrow{\mathrm{Bar}} \mathrm{coAlg}_{\mathrm{Bar}(1,\mathcal{O},1)} \xrightarrow{\mathbb{D}} \mathrm{Alg}_{\mathbb{D}\mathrm{Bar}(1,\mathcal{O},1)}?$$

No...

e.g. trivial $\mathcal{O}$-algebra A

$$A \mapsto \mathrm{Bar}(1,\mathcal{O},A) = \mathrm{Bar}(1,\mathcal{O},1)(A)$$

Divided power coalgebra $= \bigoplus_n \mathrm{Bar}(1,\mathcal{O},1)(n) \otimes_{h\Sigma_n} A^{\otimes n}$

[Francis - Gaitsgory]

$$\downarrow \mathbb{D}$$

$$\prod_n \mathbb{D}\mathrm{Bar}(1,\mathcal{O},1)(n) \otimes^{h\Sigma_n} (\mathbb{D}A)^{\otimes n}$$

[Brantner-Mathew]: For $\mathcal{O} = \mathbb{E}_\infty$-operad in $\mathrm{Mod}_{H\mathbb{F}_p}$ and A an $\mathcal{O}$-alg, $\mathbb{D}\,\mathrm{Bar}(1, \mathcal{O}, A)$ is a spectral partition Lie algebra, i.e. an algebra over the monad $\mathrm{Lie}^{\pi}_{\mathbb{F}_p, \mathbb{E}_\infty}$.

preserves filtered colim & geo. realization

KD: $\left\{ \begin{array}{l} \text{complete local} \\ \text{Noetherian} \\ \mathbb{E}_\infty\text{-}\mathbb{F}_p\text{-alg.} \end{array} \right\} \xrightarrow[\infty\text{-cats}]{\simeq} \left\{ \begin{array}{l} \text{coconnective} \\ \text{spectral partition Lie alg.} \\ \text{with degreewise finite} \\ \text{homotopy groups} \end{array} \right\}$

$\rightsquigarrow \mathrm{Moduli}_{\mathbb{F}_p, \mathbb{E}_\infty} \xrightarrow{\simeq}$ Spectral partition Lie algebras.

$$= \mathrm{Alg}_{\mathrm{Lie}^{\pi}_{\mathbb{F}_p, \mathbb{E}_\infty}}$$

This generalizes Lurie-Pridham to char > 0.

Rmk. The algebra of operations on $\mathrm{Lie}^{\pi}_{\mathbb{F}_p, \mathbb{E}_\infty}$-algs are Koszul dual to the Dyer-Lashof algebra.

§6. Chromatic Version

[Kuhn] Tate vanishing

$$(-)_{h\Sigma_n} \xrightarrow{\text{Norm}} (-)^{h\Sigma_n}$$ is an equiv. in $Sp_{T(n)}$.

Hence spectral Lie alg. $\simeq$ spectral partition Lie alg.

[Brantner] Additive operations on E-theory of spectral Lie algebras are compatible with the shifted brackets. The algebra of these operations are Koszul dual to Rezk's ring of additive operations on $\mathbb{E}_\infty$-E_n-alg.

- Recall Bousfield-Kuhn functor $\Phi : S_*^{v_n} \to Sp_{T(n)}$ which admits a left adjoint Θ.

  [Eldred-Heuts-Mathew-Meier]

  $\Phi \circ \Theta$ is a monad, and there is an equiv.

  $$S_*^{v_n} \xrightarrow{\simeq} Alg_{\Phi\circ\Theta}(Sp_{T(n)})$$

[Heuts] $\Phi \circ \Theta = \mathrm{Free}^{s\mathcal{L}}$, the monad associated to $s\mathcal{L}$ in $Sp_{T(n)}$. So $S_*^{v_n} \xrightarrow{\simeq} \mathrm{Alg}_{s\mathcal{L}}(Sp_{T(n)})$

- Generalizes Quillen's result

$$\begin{tikzcd}
 & \left\{\begin{array}{c}\text{simply-connected}\\ \text{rational pted spaces}\end{array}\right\} \arrow[dl, "\simeq"'] \arrow[dr, "\simeq"] & \\
\left\{\begin{array}{c}\text{1-connected}\\ \text{cocom. coalg}/\mathbb{Q}\end{array}\right\} & & \left\{\begin{array}{c}\text{connected}\\ \text{dgLie alg}/\mathbb{Q}\end{array}\right\}
\end{tikzcd}$$