EXAMPLES OF GOODWILLIE CALCULUS
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Abstract. In this talk, we will discuss convergence of Goodwillie towers and see some examples. We'll start by seeing how connectivity conditions are an analog of a radius of convergence for the Goodwillie tower. Then, we'll discuss an example functor for which there is a model, due to Arone, of its nth excisive approximation. Using Arone's model, we will first show that the tower converges, then deduce a result known as Snaith splitting. Finally, we will briefly discuss a connection between Goodwillie towers and the Kahn-Priddy theorem.

Contents

1. Review

Let's recall our setting. We consider functors $F\colon \mathsf{C} \to \mathsf{D}$ where C admits finite colimits and D is differentiable, meaning that it admits finite limits and sequential colimits and that these commute. We are trying to study these functors by some analog of polynomial approximation.

Definition 1.1. *A functor $F\colon \mathsf{C} \to \mathsf{D}$ is* n-excisive *if it takes strongly co-Cartesian $n+1$-cubes in C to Cartesian $n+1$-cubes in D.*

We've seen how n-excisive functors can play the role of degree n polynomials. Goodwillie showed that under the above conditions for C, D, there is a left adjoint to the inclusion of n-excisive functors.

Definition 1.2. *Let P_n be the left adjoint to the inclusion*

$$\mathrm{Exc}^n(\mathsf{C}, \mathsf{D}) \hookrightarrow \mathrm{Fun}(\mathsf{C}, \mathsf{D}) \quad (P_n\colon \mathrm{Fun}(\mathsf{C}, \mathsf{D}) \to \mathrm{Exc}^n(\mathsf{C}, \mathsf{D}))$$

We saw that P_n satisfies a couple of conditions that we would expect for a polynomial approximation:

- $P_n^2 = P_n$
- $P_m(-) \simeq P_m(P_n(-))$ for $m \leq n$.

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Lucy told us about the fibers of the maps $P_nF \to P_{n-1}F$ in her talk last week. Such a fiber is denoted D_nF and called the *nth homogeneous layer*. Each fiber D_nF is n-excisive and satisfies $P_{n-1}D_nF(X) \simeq *$ for all $X \in \mathsf{C}$. Out of this information, one can form a Taylor tower or *Goodwillie tower*:

$$\begin{array}{ccccccccc} F & \longrightarrow & \dots & \longrightarrow & P_nF & \longrightarrow & P_{n-1}F & \longrightarrow & \dots \\ & & & & \uparrow & & & & \\ & & & & D_nF & & & & \end{array}$$

However, we noted that this tower does not always converge. We will discuss convergence conditions in the next section.

Finally, in previous talks we have also seen the following formula for the nth homogeneous layer:

$$D_nF\colon X \mapsto \Omega^\infty(\partial_n F \wedge X^{\wedge n})_{h\Sigma_n} \tag{1}$$

for some $\partial_n F \in \mathsf{Sp}$. We call the coefficient $\partial_n F$ a *Goodwillie derivative*; we will see examples of $\partial_n F$ for particular functors in sections 3 and 4.

2. Convergence

Questions of convergence involve the notion of analytic functors, whose precise definition we will neglect to give in this talk. Instead, we will give some intuition for the properties an n-analytic functor should have, and show that, given those properties, we have convergence of the Goodwillie tower on appropriate objects. For the details on analytic functors, see [G92].

"Definition" 2.1. *A functor F is* n-analytic *when it's stably m-excisive for all m where connectivity estimates depend linearly on m with slope n.* [AC]

What we want to see is that n-analytic functors satisfy the following condition:

Claim 2.2. *If F is n-analytic, then the Goodwillie tower for F converges strongly on n-connected objects X.*

In our analogy, we can begin to see that the connectedness of objects on which the tower converges plays a role similar to the radius of convergence of a power series. For this reason, n as in the claim may be referred to as a *radius of convergence* for the functor F.

To get a handle on the convergence of the tower, we will deduce a vanishing line for a certain spectral sequence: the Goodwillie tower is made up of a sequence of fibrations, of which we can take homotopy and then sum together to form an exact couple, from which we can form a spectral sequence. This sequence is an example of a Bousfield-Kan spectral sequence.

Definition 2.3. *The* Goodwillie spectral sequence *has the following signature:*

$$E^1_{p,q} = \pi_p D_q F(X) \implies \pi_{p-q} P_\infty F(X).$$

For our purposes, we won't need any fancy spectral sequence computation techniques- we will find that under certain conditions, the E^1-page has a vanishing line of positive slope, which will give us strong convergence as desired. To see what those conditions are, we have to assume the following lemma.

Lemma 2.4. *If F is n-analytic and X is k-connected for $k > n$, then $F(X) \to P_qF(X)$ is at least $(d + k + q(k - n))$-connected for some d.*

We will see what d is in a later example. For now, let's just see that these conditions guarantee a vanishing line as promised. Assuming F is n-analytic, we have that $F(X) \to P_qF(X)$ is at least $(d+k+q(k-n))$-connected, while $F(X) \to P_{q-1}F(X)$ is at least $(d+k+(q-1)(k-n))$-connected, meaning that if we look at the long exact sequence in homotopy for the fibration $D_qF \to P_qF(X) \to$

$P_{q-1}F(X)$, we'll see that $D_qF(X) \to F(X)$ is itself at least $(d+k+(q-1)(k-n))$-connected. When we plug this information into the spectral sequence, we see that $E^1_{p,q} = \pi_p D_q F(X) = 0$ for

$$p \leq d + k + (q-1)(k-n).$$

Solving for q, we indeed get a vanishing line of positive slope:

$$q \geq \left(\frac{1}{k-n}\right) p - \frac{d+k}{k-n} + 1.$$

3. Arone's Model

Let's see the vanishing line in action. Let K be a finite CW complex, and consider the functor

$$\begin{aligned} F \colon \mathsf{Top}_* &\to \mathsf{Sp} \\ X &\mapsto \Sigma^\infty \mathrm{Maps}_{\mathsf{Top}_*}(K, X). \end{aligned}$$

Going forward, when we write F we will mean this functor. We'll use the results of previous section to prove the following theorem of Goodwillie.

Theorem 3.1 (Goodwillie)**.** *The tower for F converges with radius $n = \dim K$.*

proof strategy:

- use Arone's model for P_qF to deduce a model for D_qF
- determine the connectivity of D_qF
- use the vanishing line.

To show this, we will use Arone's model for the nth polynomial approximation, as described in [A]. First, we need to set some notation. Let ε be the category whose objects are finite sets $\underline{n} = \{1, \ldots, n\}$ and whose morphisms are surjections. Let ε_d be the full subcategory whose objects are finite sets of cardinality not exceeding d. For fixed d, given $X \in \mathsf{Top}_*$, define a functor

$$\begin{aligned} X^\wedge \colon \varepsilon_d^{op} &\to \mathsf{Top}_* \to \mathsf{Sp} \\ \underline{n} &\mapsto X^{\wedge n} \mapsto \Sigma^\infty X^{\wedge n}. \end{aligned}$$

Theorem 3.2 (Arone)**.** *There is an identification*

$$P_dF(X) \simeq \mathrm{Map}_{\mathsf{Fun}(\varepsilon_d^{op}, \mathsf{Sp})}(\Sigma^\infty K^\wedge, \Sigma^\infty X^\wedge).$$

Given this theorem, we can write out a similar model for the fiber.

Claim 3.3. *There is a homotopy pullback in spectra*

$$\begin{array}{ccc} P_dF(X) & \longrightarrow & \mathrm{Map}^{\Sigma_d}_{\mathsf{Sp}}(\Sigma^\infty K^{\wedge d}, \Sigma^\infty X^{\wedge d}) \\ \downarrow & & \downarrow \\ P_{d-1}F(X) & \longrightarrow & \mathrm{Map}^{\Sigma_d}_{\mathsf{Sp}}(\Sigma^\infty \delta_d K, \Sigma^\infty X^{\wedge d}) \end{array}$$

where $\delta_d K = \{(k_1, \ldots, k_d) \in K^d \mid \exists i \neq j,\ k_i = k_j\}$ is the fat diagonal and $\Sigma_d \subset \varepsilon_d$ is the subcategory whose objects are sets of cardinality exactly d and whose morphisms are permutations.

proof idea: Write in Arone's model on the LHS:

$$\begin{array}{ccc} \mathrm{Map}_{\mathsf{Fun}(\varepsilon_d^{op}, \mathsf{Sp})}(\Sigma^\infty K^\wedge, \Sigma^\infty X^\wedge) & \longrightarrow & \mathrm{Map}^{\Sigma_d}_{\mathsf{Sp}}(\Sigma^\infty K^{\wedge d}, \Sigma^\infty X^{\wedge d}) \\ \downarrow & & \downarrow \psi \\ \mathrm{Map}_{\mathsf{Fun}(\varepsilon_{d-1}^{op}, \mathsf{Sp})}(\Sigma^\infty K^\wedge, \Sigma^\infty X^\wedge) & \xrightarrow{\varphi} & \mathrm{Map}^{\Sigma_d}_{\mathsf{Sp}}(\Sigma^\infty \delta_d K, \Sigma^\infty X^{\wedge d}) \end{array}$$

The data of a natural transformation $f' \in \mathrm{Map}_{\mathsf{Fun}(\varepsilon_d^{op}, \mathsf{Sp})}(\Sigma^\infty K^\wedge, \Sigma^\infty X^\wedge)$ includes a commuting diagram of maps

$$
\begin{array}{ccc}
\Sigma^\infty K^{\wedge d} & \xrightarrow{f'_d} & \Sigma^\infty X^{\wedge d} \\
\uparrow & & \uparrow \\
\Sigma^\infty K^{\wedge d-1} & \xrightarrow{f'_{d-1}} & \Sigma^\infty X^{\wedge d-1} \\
\uparrow & & \uparrow \\
\vdots & & \vdots \\
\uparrow & & \uparrow \\
\Sigma^\infty K^{\wedge 1} & \xrightarrow{f'_1} & \Sigma^\infty X^{\wedge 1}
\end{array}
$$

In the alleged pullback diagram, the map ψ is the restriction of a natural transformation to just the fat diagonal, while the map φ involves duplicating components f'_i to land in the fat diagonal. We can think of natural transformations $h\colon \Sigma^\infty K^{\wedge d} \to \Sigma^\infty X^{\wedge d}$ living in the top right of the square as encoding degree d information, while natural transformations $f\colon \Sigma^\infty K^{\wedge} \to \Sigma^\infty X^{\wedge}$ in the bottom left encode information up to degree $d-1$. A natural transformation $g\colon \Sigma^\infty \delta_d K \to \Sigma^\infty X^d$ from the bottom right tells us how to glue together the information from h and f to build f'. In a diagram, the data for f' now takes the following form:

$$
\begin{array}{ccc}
\Sigma^\infty K^{\wedge d} & \xrightarrow{h} & \Sigma^\infty X^{\wedge d} \\
{\scriptstyle \psi^*}\uparrow & & {\scriptstyle \psi^*}\uparrow \\
\Sigma^\infty \delta_d K & \xrightarrow{g} & \Sigma^\infty X^{\wedge d} \\
{\scriptstyle \varphi}\uparrow & & {\scriptstyle \varphi}\uparrow \\
\Sigma^\infty K^{\wedge d-1} & \xrightarrow{f_{d-1}} & \Sigma^\infty X^{\wedge d-1} \\
\uparrow & & \uparrow \\
\Sigma^\infty K^{\wedge d-2} & \xrightarrow{f_{d-2}} & \Sigma^\infty X^{\wedge d-2} \\
\uparrow & & \uparrow \\
\vdots & & \vdots \\
\uparrow & & \uparrow \\
\Sigma^\infty K^{\wedge 1} & \xrightarrow{f_1} & \Sigma^\infty X^{\wedge 1}
\end{array}
$$

Given the claim, we can compute D_dF by computing the homotopy fiber of the RHS vertical map of the homotopy pullback square. Write $K^{(d)}$ for the quotient $K^{\wedge d}/\delta_d K$. We find that

$$
\begin{aligned}
D_dF(X) &\simeq \mathrm{Map}^{\varepsilon_d}_{\mathsf{Sp}}(\Sigma^\infty K^{(d)}, \Sigma^\infty X^{\wedge d}) \\
&\simeq \mathrm{Map}_{\mathsf{Sp}}(\Sigma^\infty K^{(d)}, \Sigma^\infty X^{\wedge d})_{h\Sigma_d}
\end{aligned}
$$

where in the second line we use that Σ_d acts freely on $K^{(d)}$, as we have quotiented out all configurations with repeated points.

Now, we have a model for D_dF, but its connectivity is not necessarily obvious. To understand that better, we can rewrite it using Spanier-Whitehead duals.

Remark 3.4. *Recall that for X a finite spectrum, its* Spanier-Whitehead dual *is given by*

$$\mathbb{D}X = \mathrm{Map}_{\mathcal{Sp}}(X, \mathbb{S}^0).$$

Morally, if we think of spectra as a linearization of spaces, we see that the SW dual takes the form of a linear dual; i.e. homomorphisms into a unit.

We can thus rewrite

$$D_d F(X) \simeq (\mathbb{D}\Sigma^\infty K^{(d)} \wedge \Sigma^\infty X^{\wedge d})_{h\Sigma_d}.$$

From formula (1), we can then recognize $\partial_d F = \mathbb{D}\Sigma^\infty K^{(d)}$.

Let's return to Goodwillie's theorem and sketch in the last two parts of the outline.

sketch proof of 3.1: We assume X is k-connected, so its bottom cell is in dimension $k+1$, and hence X^q has bottom cell in dimension $q(k+1)$. Meanwhile, we set $\dim K = n$, so $K^{(q)}$ is qn-dimensional. This is the dimension of the top cell of $K^{(q)}$, so the dimension of the bottom cell of the dual $\mathbb{D}K^{(q)}$ is $-qn$. Putting all of this together with the formula for $D_q F(X)$, we see that $D_q F(X)$ has bottom cell in dimension $q(-n+k+1)$, so is $(q(-n+k+1)+1)$-connected.

As discussed after Lemma 2.4, this leads to a vanishing line in the spectral sequence, which gives us strong convergence. Comparing with the lemma statement, we see that $d = q-1-n$ for the functor F.

4. Snaith Splitting

Next we show how to deduce Snaith splitting using the Goodwillie tower in the previous section. For some motivation, recall that for a topological space X, there is an identification of its stable homotopy groups as

$$\pi_*^s X \simeq \pi_* \Omega^\infty \Sigma^\infty X.$$

Assuming that we are interested in the LHS, it is then compelling to study the RHS. The space $\Omega\Sigma X$ is homotopy equivalent to the James construction, and more generally, if $C^{(n)}$ is the little n-cubes operad, then $\Omega^n \Sigma^n X$ is homotopy equivalent to the free $C^{(n)}$-algebra on X modulo a relation making the basepoint of X the identity, as shown by May. See, for example, [May] or [Mat] for more on this story.

Write $C^{(n)}(k)$ for the space of embeddings of k little n-cubes in a big n-cube. Then Snaith splitting is the following result.

Theorem 4.1 (Snaith)**.** *There is a splitting*

$$\Sigma^\infty \Omega^n \Sigma^n X = \bigvee_{k \geq 0} \Sigma^\infty (C^{(n)}(k)_+ \wedge X^{\wedge k})_{h\Sigma_k}.$$

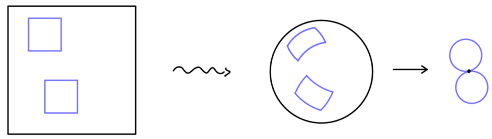
We can work toward this result using Arone's model for the functor F in the case that $K = S^n$ is a sphere and $X = \Sigma^n Y$ is an n-fold suspension. In this case,

$$\begin{aligned} D_k F(\Sigma^n Y) &\simeq (\mathbb{D}\Sigma^\infty (S^n)^{(k)} \wedge \Sigma^\infty (\Sigma^n Y)^{\wedge k})_{h\Sigma_k} \\ &\simeq (\mathbb{D}\Sigma^\infty (S^n)^{(k)} \wedge \mathbb{S}^{nk} \wedge \Sigma^\infty Y^{\wedge k})_{h\Sigma_k}. \end{aligned}$$

To relate this to Snaith's result, we need to recognize $\mathbb{D}\Sigma^\infty (S^n)^{(k)} \wedge \mathbb{S}^{nk}$ as $\Sigma^\infty_+ C^{(n)}(k)$. To that end, consider that we have a map

$$\alpha(n,k) \colon C^{(n)}(k) \to \mathrm{Maps}_{\mathsf{Top}_*}(S^n, \bigvee_k S^n)$$
$$\coprod_k I^n \hookrightarrow I^n \rightsquigarrow S^n \to \bigvee_k S^n$$

given by using the Pontryagin-Thom collapse. Just as a reminder, a rough sketch of this map for $n = 2, k = 2$ is



Using an adjoint to $\alpha(n,1)$ and forming a composite, one can define a map

$$\delta(n,k)\colon C^{(n)}(k)_+ \wedge S^{n(d)} \to S^{nk}$$

going the other direction. We will not go through the details, but quote the following result from [AK].

Theorem 4.2 (Ahearn-Kuhn)**.** *The map $\delta(n,k)$ induces a Σ_k-equivariant equivalence*

$$\Sigma^\infty_+ C^{(n)}(k) \simeq \mathrm{Map}_{\mathcal{S}p}(\Sigma^\infty S^{n(k)}, S^{nk}).$$

To deduce Snaith splitting, we will need to assume the following fact.

Fact 4.3. *When $K = S^n$, $X = \Sigma^n Y$, the Goodwillie tower for F strongly splits.*

This begs the question of whether showing this fact is easier than taking Snaith's approach, but from here, it is quick to deduce the result:

$$\begin{aligned}\Sigma^\infty\Omega^n\Sigma^n Y &= F(\Sigma^n Y)\\ &\simeq \bigvee_{k>0} D_k F(\Sigma^n Y)\\ &\simeq \bigvee_{k>0} \Sigma^\infty (C^{(n)}(k)_+ \wedge X^{\wedge k})_{h\Sigma_k}.\end{aligned}$$

What happens in the limit as $n \to \infty$? Since $\mathrm{hocolim}_{n\to\infty} C^{(n)}(k)_+$ is a model for $E\Sigma_{k+} \sim_{weq} S^0$ and $\mathrm{hocolim}_{n\to\infty}\Sigma^{-n}\Sigma^\infty X_n \to X$ is an equivalence for X_n the nth space of the spectrum X, we can apply the above to this case.

Consider the functor $G\colon X \mapsto \Sigma^\infty\Omega^\infty X$. This is the identity on spectra and we'll hear more about it later in the seminar. Since smashing with S^0 is the identity, we see that $D_d G(X) \simeq X^{\wedge d}_{h\Sigma_d}$. In fact, we know more.

Fact 4.4. *For X a 0-connected suspension spectrum, the Goodwillie tower converges, and*

$$\Sigma^\infty\Omega^\infty X = X \oplus X^{\wedge 2}_{h\Sigma_2} \oplus X^{\wedge 3}_{h\Sigma_3} \oplus \dots$$

This means that applying the functor $\Sigma^\infty\Omega^\infty$ is like taking the group ring $\mathbb{S}[X]$. Going back to our power series analogy, we can recognize the RHS as encoding polynomials: it features different powers of X and enforces commutativity of multiplication by taking homotopy orbits.

5. The Kahn-Priddy Theorem

We begin with the observation that there's a unique map $\varepsilon\colon \mathbb{R}P^\infty \to *$. It's not very interesting. If we take Σ^∞_+, we get

$$\Sigma^\infty_+\mathbb{R}P^\infty \xrightarrow{\varepsilon} \mathbb{S},$$

which is also not very interesting. However, on the level of spectra, there is actually another map,

$$\Sigma^\infty_+\mathbb{R}P^\infty \xrightarrow{\mathrm{tr}} \mathbb{S},$$

called the *transfer*. It turns out that this map *is* interesting.

Theorem 5.1 (Kahn-Priddy)**.** *At the prime 2, the transfer map is surjective on homotopy.*

Note that $\Sigma^\infty_+ \mathbb{R}P^\infty$ splits into $\Sigma^\infty \mathbb{R}P^\infty \oplus \mathbb{S}$. On the first component, ε is trivial, but tr is interesting and surjective. On the second component, ε just projects, while tr is a kind of multiplication by 2. Altogether, since $\mathbb{R}P^\infty \simeq BC_2$, the transfer is surjective on homotopy at the prime 2. Hence we assume that we are working 2-locally for the rest of this section.

We will neglect a discussion of the construction of the transfer map in favor of getting back to Goodwillie calculus. In the previous section, we saw how the Goodwillie tower for the functor G splits on 0-connected suspension spectra. However, the tower gives us a way to understand G on a wider range of objects. Substituting the formula for the fibers, we have

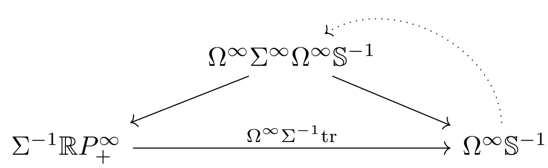
$$\begin{array}{ccc} X & \longmapsto & \Sigma^\infty \Omega^\infty X \\ & & \downarrow \\ & & \vdots \\ & & \downarrow \\ X^{\otimes 3}_{h\Sigma_3} & \longrightarrow & P_3 G \\ & & \downarrow \\ X^{\otimes 2}_{hC_2} & \longrightarrow & P_2 G \\ & & \downarrow \\ & & X. \end{array}$$

To see the relationship to the transfer map, we plug in $X = \mathbb{S}^{-1}$. It turns out that when we take C_2 homotopy orbits of $\mathbb{S}^{-1} \otimes \mathbb{S}^{-1}$, where the C_2 action is given by swapping factors, we get $\Sigma^{-1}\mathbb{R}P^\infty_{-1}$ for $D_2 G(\mathbb{S}^{-1})$. Meanwhile, $P_2 G(\mathbb{S}^{-1})$ may be identified with $\Sigma^{-1}\mathbb{R}P^\infty_+$. Finally, we claim that the vertical map $P_2 G(\mathbb{S}^{-1}) \to \mathbb{S}^{-1}$ is actually a desuspension of the transfer map.

Altogether, we have claimed that the bottom of the Goodwillie tower for G becomes

$$\begin{array}{ccc} & & \Sigma^\infty \Omega^\infty \mathbb{S}^{-1} \\ & & \downarrow \\ & & \vdots \\ & & \downarrow \\ \Sigma^{-1}\mathbb{R}P^\infty_{-1} & \longrightarrow & \Sigma^{-1}\mathbb{R}P^\infty_+ \\ & & \downarrow {\scriptstyle \Sigma^{-1}\mathrm{tr}} \\ & & S^{-1} \end{array}$$

Admitting these claims, how can we deduce something close to the Kahn-Priddy theorem? Applying Ω^∞ to the top and bottom portions of the tower gives



We have drawn a section in this diagram splitting the map $\Omega^\infty\Sigma^\infty\Omega^\infty\mathbb{S}^{-1} \to \Omega^\infty\mathbb{S}^{-1}$. This section comes from the unit of the suspension-loops adjunction applied to the identity map id: $Z \mapsto \Sigma^\infty\Omega^\infty Z$. Composed with the left arrow, this section will split the desuspension of the transfer. This amounts to "one suspension away" from the Kahn-Priddy theorem at the prime 2, since $\pi_*(\Omega^\infty X) = \pi_*(X)$. For details, see [K04] Appendix.

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