Nilpotence detecting Lubin-Tate theories at heights 0, 1

We will prove the following result, which implies Nullstellensatz at height 1

<u>Thm</u> Let $R \in \mathrm{CAlg}(\mathrm{Sp}_{T(1)})$. $\exists$ a nilpotence detecting map

$$R \to E(K) \quad \text{for} \quad K \text{ a reduced krull dim } 0 \text{ ring.}$$

first we'll prove the analogous result at height 0

<u>Thm</u> Let $R \in \mathrm{CAlg}(\underset{\mathrm{Sp}_{\mathbb{Q}}}{\overset{\|}{\mathrm{Sp}_{T(0)}}})$. $\exists$ nilp det map

$$R \to K[t_2^{\pm}] \quad \text{for some } K \text{ reduced krull dim } 0.$$

<u>Idea</u> keep modifying R to make it closer to $K[t_2^{\pm}]$ while at every step making sure the new object detects nilpotence.

keep doing until new ring looks like what you want

Nilpotence detecting maps are closed under
- (transfinite) compositions
- cobasechange

so a small object argument shows that for any collection of nilpotence detecting maps $\alpha_i : A_i \to B_i$,

$\exists \; R \to S$ detecting nilpotence s.t

$$\begin{tikzcd} A_i \arrow[r] \arrow[d] & S \\ B_i \arrow[ur, dashed] & \end{tikzcd} \quad \text{ie} \quad \alpha_i \perp S.$$

"right lifting property"

We will choose 3 maps

$$\alpha_1 = \mathbb{Q} \to \mathbb{Q}[x_2^{\pm}]$$

$$\alpha_2 = \mathbb{Q}[\varepsilon_{-1}] \to \mathbb{Q}$$

$$\alpha_3 = \mathbb{Q}[t] \to \mathbb{Q} \times \mathbb{Q}[t^{\pm}]$$

α_1 detects nilpotence...

α_2 does too; it is

$$\mathbb{Q} \text{ modules w/ loc. nilpotent end} \xrightarrow{\text{forget}} \mathbb{Q}\text{-modules}$$

α_3 does too: if $X \xrightarrow{f} Y$ has $f^{\otimes n}[t^{-1}] = 0$

$f^{\otimes m}/t = 0$

$\Rightarrow f^{\otimes n} t^k = 0$

$k \gg 0$

$\Rightarrow f^{\otimes \alpha} = 0$

$\alpha \gg 0$.

$S \perp \alpha_1 \Rightarrow S$ is 2-periodic

$+ \alpha_2 \Rightarrow S$ is even

$+ \alpha_3 \Rightarrow \pi_0 S$ reduced Krull dim 0.

so we have $R \to S$ detecting nilpotence w/ S satisfying those.

$S \to \prod_{x \in \operatorname{Spec} S} S_x$ detects nilpotence. Suffices to show

$S_x \cong \pi_0 S_x[t_2^{\pm}]$ so that

$\prod S_x \cong (\prod \pi_0 S_x)[t_2^{\pm}]$ and we're done.

$\pi_0 S_{x^\cdot} = \pi_0 S_x[t_2^{\pm}]$ which is formally smooth, bc S_x is a field of char 0.
so there is no obstruction to building a section

$\pi_0 S_x \underset{\text{section}}{\overset{\longleftarrow}{\dashrightarrow}} \tau_{\geq 0} S_x$ which composing w/ map to S_x
and tensoring w/ $\mathbb{Q}[t^{\pm}] \to S_x$ gives an equivalence.

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height 1. $E(\mathbb{F}_p) \cong KU_p^\wedge$

$\mathbb{S}_{T(1)} \to \overset{E}{\overset{\shortparallel}{E(\mathbb{F}_p)}}/p$ detects nilpotence by nilpotence thm.

$\Rightarrow$ We can assume WLOG $R \in \mathrm{CAlg}(\mathrm{Mod}(E)_{K(1)})$
$\overset{\shortparallel}{\mathrm{CAlg}_E}$

<u>Thm</u> $\exists$ a fully faithful functor

$E(-): \mathrm{Perf}_{\mathbb{F}_p} \to \mathrm{CAlg}_E$

w/ right adjoint given by

$R \to (\pi_0 R/p)^\flat = R^\flat = \lim \cdots \pi_0 R/p \xrightarrow{\text{Frob}} \pi_0 R/p \xrightarrow{\text{Frob}} \pi_0 R/p$

<u>Pf</u> $\mathrm{Perf}_{\mathbb{F}_p}$ has cotangent complex vanishing
so giving a map amounts to something on π_0/p. $\square$

The counit is a map

$E(R^\flat) \to R$. R is $E(A) \Leftrightarrow$ this is an iso.

We'll run a small abs arg & use the following to see
the result is what we want:

Suppose $R \in \mathrm{CAlg}_{\hat{E}}$ is

1) even
2) $E(R^\flat) \to R$ is surj on π_0
3) $R^\flat$ is Krull dim 0 reduced.

$\Rightarrow R \cong E(R^\flat)$

Pf $3 \Rightarrow E(R^\flat) \to R$ is inj on π_0 bc $R^\flat \to \prod_{\mathfrak{p}} R/\mathfrak{p}$ cannot have a kernel or else an idempotent would split off.
Thus $E(R^\flat) \xrightarrow{\sim} R$.

So we use 3 maps

1) $E\{x_1\} \to E$ $\Rightarrow$ 1) if $\perp$

3) $E[t^{1/p^\infty}] \to E[t^{\pm 1/p^\infty}] \times E$ $\Rightarrow$ 3) if $\perp$

Need to check 1) detects nilpotence.

$$\begin{tikzcd} E\{x_{-1}\} \arrow[r] \arrow[d, "x_{-1} \mapsto 0"'] & E \arrow[d] \\ R \arrow[r] & R\{x_0\} \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

this detects nilpotence.

fib $(R \to R\{x_0\})$ is a free R-module in odd degrees bc free $K(1)$-local $\mathbb{E}_\infty$ ring is <u>even</u>.

More generally if x_{-1} isn't sent to zero (eg id map)

fib $(R \to \text{pushout})$ has a filtration w/ associated graded free R-module in odd degrees.

In this situation, you can show

$[\mathrm{fib}\, E\{x\} \to E)^{\otimes 2} \longrightarrow E\{x\}$ is a phantom map.

$\Rightarrow$ detects nilpotence.

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finally need to construct a map forcing 2).

$$A^{\#} = \operatorname{colim} A \xrightarrow{\mathrm{Frob}} A \xrightarrow{\mathrm{Frob}} A \xrightarrow{\mathrm{Frob}} \cdots$$

$\mathbb{F}_p\{x\} =$ free δ-ring on x/p.

To force 2) we produce a map g

$g : E\{x\} \longrightarrow E(\mathbb{F}_p\{x\}^{\#})$ such that on

π_0/p it is

$\mathbb{F}_p\{x\} \longrightarrow \mathbb{F}_p\{x\}^{\#}$. If $g \perp S$,

$\pi_0 S/p$ has surj Frobenius $\Rightarrow E(R^b) \to R$ surj on π_0.

Such a map is easily seen to detect nilpotence;

mod p, $\mathbb{F}_p\{x\}^{\#}$ is a free module over $\mathbb{F}_p\{x\}$

(it is p-completely faithfully flat)

π_0 is naturally a δ-ring, and $\pi_0(E(A))$ is the <u>cofree</u> δ-ring on A.

$$
\begin{tikzcd}
{[E\{x\}, E(\mathbb{F}_p\{x\}^\#)]_{\mathrm{CAlg}_E}} \arrow[dd, bend right, "\pi_0"'] \arrow[d, phantom, "\parallel"] \\
\pi_0 E(\mathbb{F}_p\{x\}^\#) = W(\mathbb{F}_p\{x\}^\#) \arrow[d, phantom, "\parallel"] \\
\mathrm{Hom}_{\delta\text{-ring}}(\mathbb{Z}_p\{x\}, W(\mathbb{F}_p\{x\}^\#)) \arrow[d, phantom, "\parallel \leftarrow \text{cofreeness}"] \\
\mathrm{Hom}_{\mathrm{ring}}(\mathbb{Z}_p\{x\}, \mathbb{F}_p\{x\}^\#) \arrow[d, phantom, "\parallel"] \\
\mathrm{Hom}_{\mathrm{perf}_{\mathbb{F}_p}}(\mathbb{F}_p\{x\}^\#, \mathbb{F}_p\{x\}^\#) \ \mathrm{id} \arrow[uuuu, bend right, "g"']
\end{tikzcd}
$$

so we're done.

We need this cofreeness result at higher heights, which is the main technical ingredient in the proof. Proving the cofreeness result will involve understand operations on $K(n)$-local E_n-algebras well.

- Eunice's talk: $\mathbb{T}$-algebras generalize δ-rings to higher heights.
- Natalie's talk: Nilpotence detection
- Arpon's talk: explaining proof at higher height.
- David/Nat/Adela's talks build up to proving cofreeness.