Power Operations & Rezk's Π-algebras

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Outline

① power operations on E-homology

- motivation : $H\mathbb{F}_p$ (Dyer-Lashof)

things in grey are notes to myself --- can ignore

② power operations on $E = E_n$

- construct Π
- approximation map

③ additive operations ($\longleftrightarrow$ ker of transfer)

④ dual perspectives

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1. Power operations

$E \in Sp$

Power operations on E-cohomology = nat trans $E^i(-) \longrightarrow E^j(-)$ of functors $hSp \longrightarrow Sets$

If E E_∞-ring, can find this by finding nat trans $\pi_i(-) \longrightarrow \pi_j(-)$ of functors $hAlg_E \longrightarrow Sets$

π_i corepresented by $\mathbb{P}(\Sigma^i E)$ in $hAlg_E$

$\mathbb{P}$ = free comm E-alg monad on $hMod_E$ $\left(\mathbb{P}(X) = \bigoplus_{m \geq 0} \underbrace{\left(X^{\otimes_E m}\right)_{h\Sigma_m}}_{\mathbb{P}_m(X)}\right)$

$\mathbb{P} = \bigoplus_{m \geq 0} \mathbb{P}_m$

Yoneda : $\left\{\begin{array}{l}\text{power ops } \pi_i \longrightarrow \pi_j \\ hAlg_E \longrightarrow Sets\end{array}\right\} \longleftrightarrow \pi_j \mathbb{P}(\Sigma^i E)$

$\cup$ $\qquad\qquad\qquad\qquad \cup$

primitive elements

$\left\{\begin{array}{l}\text{additive ops } \pi_i \longrightarrow \pi_j \\ hAlg_E \longrightarrow Ab\end{array}\right\} \longleftrightarrow = eq\left(\pi_j \mathbb{P}(\Sigma^i E) \underset{\mathbb{P}(i_1)+\mathbb{P}(i_2)}{\overset{\mathbb{P}(i_1+i_2)}{\rightrightarrows}} \pi_j \mathbb{P}(\Sigma^i E \oplus \Sigma^i E)\right)$

Want to construct algebraic approx $\mathbb{T}$ of $\mathbb{P}$

- $\underline{E = H\mathbb{F}_p}$

$$\begin{tikzcd} h\mathrm{Mod}_{H\mathbb{F}_p} \arrow[r, "\pi_*", "\sim"'] \arrow[d, "\mathbb{P}"'] \arrow[dr, phantom, "\circ"] & \mathrm{Mod}_{\mathbb{F}_p} \arrow[d, "\mathbb{T}"] \\ h\mathrm{Mod}_{H\mathbb{F}_p} \arrow[r, "\pi_*", "\sim"'] & \mathrm{Mod}_{\mathbb{F}_p} \end{tikzcd}$$

(= discrete graded $\mathbb{F}_p$-modules)

π_* sym. monoidal equiv $\Rightarrow$ get monad $\mathbb{T}$ on $\mathrm{Mod}_{\mathbb{F}_p}$

$\alpha : \mathbb{T}\pi_*(-) \longrightarrow \pi_*\mathbb{P}(-)$ nat iso

Power operations are parametrised by $\pi_j \mathbb{P}(\Sigma^i E) \simeq \mathbb{T}(\mathbb{F}_p[i])_j$

Additive operations = primitive elements in $\mathbb{T}(\mathbb{F}_p[i])_j$

$$= eq\left(\mathbb{T}(\mathbb{F}_p[i])_j \overset{\mathbb{T}(i_1+i_2)}{\underset{\mathbb{T}(i_1)+\mathbb{T}(i_2)}{\rightrightarrows}} \mathbb{T}(\mathbb{F}_p[i] \oplus \mathbb{F}_p[i])_j \right)$$

So, can describe power ops entirely algebraically using $\mathbb{T}$

$\mathrm{Alg}_{\mathbb{T}}$ = $\mathrm{Alg}_{\mathbb{F}_p}$ with Dyer-Lashof operations

2. E = Morava E-theory

Fix prime p, height n, k perfect field of char p

$E = E(k)$, $E_* = \pi_* E \simeq \underbrace{W(k)[[u_1, \dots, u_{n-1}]]}_{\deg 0}\underbrace{[u^{\pm 1}]}_{\deg 2}$, $E_0 = \pi_0 E$

Will only consider operations $\pi_0 \longrightarrow \pi_0$

(Since $\pi_*(E)$ is 2-periodic, all other ops can be obtained by some twisting operations.)

Ishan: telescope conjecture is true on COCT ⇒ Morava E-thy, so for our case $K(n) \simeq T(n)$.

Problems:

- $$\begin{tikzcd} h\mathrm{Mod}_E \arrow[r, "\not\simeq", "\pi_*"'] & \mathrm{Mod}_{E_*} \simeq \mathrm{Mod}_{E_0} \\ h\mathrm{Mod}_E^{\mathrm{free}} \arrow[u, phantom, "\cup"] \arrow[r, "\sim", "\pi_*"'] & \mathrm{Mod}_{E_*}^{\mathrm{free}} \arrow[u, phantom, "\cup"] \end{tikzcd}$$

  but

  $\mathbb{Z}$-graded E_0-mods

  $\mathbb{Z}/2$-graded E_0-mods

  (Re209, 2.10)

- For M free E-mod, $\mathbb{P}(M)$ not necessarily free

  but if M finite free (f.g.) $\Rightarrow L_{K(n)}\mathbb{P}_m(M)$ fin free

  but $L_{K(n)}\mathbb{P}(M)$ not necessarily free

  ($L_{K(n)}$ doesn't preserve coproducts)

$\Rightarrow$ On fin free, set $\mathbb{T}_m \pi_* M := \pi_* L_{K(n)} \mathbb{P}_m(M)$

$$\begin{tikzcd} h\mathrm{Mod}_E^{\mathrm{fin\,free}} \arrow[r, hook, "i"] \arrow[d, "\pi_*"', "\wr"] & h\mathrm{Mod}_E \arrow[r, "\pi_* L_{K(n)} \mathbb{P}_m"] \arrow[d, "\pi_*"] & \mathrm{Mod}_{E_*} \\ \mathrm{Mod}_{E_*}^{\mathrm{fin,free}} \arrow[r, hook, "j"'] \arrow[ru, phantom, "\circ"] & \mathrm{Mod}_{E_*} \arrow[ru, dashed] & \end{tikzcd}$$

$$\boxed{\begin{gathered} \mathbb{T}_m := \mathrm{LKan}_{\pi_* i}\, \pi_* L_{K(n)} \mathbb{P}_m i \\ \mathbb{T} = \bigoplus_{m \geq 0} \mathbb{T}_m \end{gathered}}$$

$\exists \quad \alpha_m : T_m \pi_* \longrightarrow \pi_* L_{K(n)} \mathbb{P}_m$

$\rightsquigarrow \boxed{\underline{\text{approximation map}} \quad \alpha : T\pi_* = \bigoplus_{m \geq 0} T_m \pi_* \longrightarrow \pi_* L_{K(n)} \mathbb{P}}$

$$\begin{tikzcd} T\pi_* = \bigoplus_m T_m \pi_* \arrow[r] & \bigoplus_m \pi_* L_{K(n)} \mathbb{P}_m = \pi_* \bigoplus_m L_{K(n)} \mathbb{P}_m \arrow[d] \\ & \pi_* L_{K(n)} \mathbb{P} \end{tikzcd}$$

$\mathbb{P}_m \longrightarrow \mathbb{P} \implies L_{K(n)} \mathbb{P}_m \longrightarrow L_{K(n)} \mathbb{P}$

$\implies \bigoplus_m L_{K(n)} \mathbb{P}_m \longrightarrow L_{K(n)} \mathbb{P}$

⚠ α_m iso on finite frees, but α is not

For free, $(T\pi_*)^{\wedge}_{\mathfrak{m}} \xrightarrow{\sim} \pi_* L_{K(n)} \mathbb{P}$, $\mathfrak{m} \subset \pi_0 E$ max ideal.

Actually $\rightsquigarrow$ $\hat{\alpha} : L_0 T\pi_* \longrightarrow \pi_* L_{K(n)} \mathbb{P}$ iso on flats

derived completion

$\cong$ — for frees

$(T\pi_*)^{\wedge}_{\mathfrak{m}}$

approx map $\implies \exists$ lift

$$\begin{tikzcd} \mathrm{Alg}_{\mathbb{P}} \arrow[r, dashed, "\pi_* L_{K(n)}"] \arrow[d] & \mathrm{Alg}_{T} \arrow[d] \\ h\mathrm{Mod}_E \arrow[r, "\pi_* L_{K(n)}"] & \mathrm{Mod}_{E_*} \end{tikzcd}$$

③ Additive operations

Γ = ring of additive operations on $\mathrm{Alg}_{\mathbb{T}}$ in deg 0 = $\mathrm{End}(U^0)$

$$U^0 : \mathrm{Alg}_{\mathbb{T}} \xrightarrow{\text{deg 0 part}} \mathrm{Ab}$$
$$A \longmapsto A_0$$

$\mathrm{Alg}_{\mathbb{T}} \xrightarrow{U^0} \mathrm{Ab} \longrightarrow \mathrm{Sets}$ is corep by $\mathbb{T}(E_0)$

$$\Rightarrow \mathrm{End}_{\mathrm{Sets}}(U^0) = \mathrm{Hom}_{\mathrm{Alg}_{\mathbb{T}}}(\mathbb{T}(E_0), \mathbb{T}(E_0)) \simeq \mathbb{T}(E_0)_0$$

$$\mathrm{End}(U^0) = \text{deg 0 primitives in } \mathbb{T}(E_0)$$
$$= \mathrm{eq}\left(\mathbb{T}(E_0)_0 \overset{\mathbb{T}(i_1+i_2)}{\underset{\mathbb{T}(i_1)+\mathbb{T}(i_2)}{\rightrightarrows}} \mathbb{T}(E_0 \oplus E_0)_0 \right)$$

let $\Delta^+ : \mathbb{T}(E_0) \xrightarrow{\mathbb{T}\Delta} \mathbb{T}(E_0 \oplus E_0) \xrightarrow{\sim} \mathbb{T}(E_0) \otimes \mathbb{T}(E_0)$

$$\Rightarrow \mathrm{End}(U^0) = \{ f \in \mathbb{T}(E_0)_0 \mid \Delta^+(f) = f \otimes 1 + 1 \otimes f \}$$

$$\mathbb{T}_m(E_0) = \pi_0 L_{K(n)} \mathbb{P}_m(E) = \pi_0 L_{K(n)} E_{h\Sigma_m} \simeq E_0^{\wedge} B\Sigma_m$$

$$E_0^{\wedge} X := \pi_0 L_{K(n)}(E \otimes X)$$

$$\mathbb{T}_m(E_0 \oplus E_0) = \pi_0 L_{K(n)} (E_0 \oplus E_0)^{\otimes m}_{h\Sigma_m} \simeq \bigoplus_{0 \le j \le m} E_0^{\wedge} B(\Sigma_j \times \Sigma_{m-j})$$

$$\Rightarrow \Gamma = \mathrm{End}(U^0) = \bigoplus_{m \ge 0} \ker\left(E_0^{\wedge} B\Sigma_m \xrightarrow{\text{transfer}} \bigoplus_{0<j<m} E_0^{\wedge} B(\Sigma_j \times \Sigma_{m-j}) \right)$$
$$= \bigoplus_{k \ge 0} \ker\left(E_0^{\wedge} B\Sigma_{p^k} \longrightarrow \bigoplus_{0<j<p^k} E_0^{\wedge} B(\Sigma_j \times \Sigma_{p^k-j}) \right)$$

$\pi_0 : h\mathrm{Alg}_{E, K(n)} \longrightarrow Ab$

$\mathrm{End}(\pi_0) =$ ring of additive operations in deg 0

$$\boxed{\text{Prop}: \ \Gamma^{\wedge}_{m} \simeq \mathrm{End}(\pi_0)}$$

Pf: π_0 corep by $L_{K(n)} \mathbb{P}(E)$

$\Rightarrow \mathrm{End}_{Sets}(\pi_0) = \pi_0 L_{K(n)} \mathbb{P}(E)$

$$\mathrm{End}(\pi_0) = eq\left(\pi_0 L_{K(n)} \mathbb{P}(E) \rightrightarrows \pi_0 L_{K(n)} \mathbb{P}(E \oplus E)\right)$$

$0 \longrightarrow \mathrm{End}(\pi_0) \longrightarrow \pi_0 L_{K(n)} \mathbb{P}(E) \longrightarrow \pi_0 L_{K(n)} \mathbb{P}(E \oplus E)$ exact

$0 \longrightarrow \Gamma \xrightarrow{i} \mathbb{T}(E_0)_0 \xrightarrow[\mathbb{T}(i_1+i_2) - \mathbb{T}(i_1) - \mathbb{T}(i_2)]{j} \mathbb{T}(E_* \oplus E_*)_0$ exact

Claim: remains exact after completing. (won't prove)

Assuming claim,

$$\begin{tikzcd}
0 \arrow[r] & \Gamma^{\wedge}_{m} \arrow[r, "i"] \arrow[d, dashed, "\Rightarrow \wr"'] & \mathbb{T}(E_0)_0{}^{\wedge}_{m} \arrow[r, "j"] \arrow[d, "\wr", "\hat{\alpha}"'] & \mathbb{T}(E_* \oplus E_*)_0{}^{\wedge}_{m} \arrow[d, "\wr", "\hat{\alpha}"'] \\
0 \arrow[r] & \mathrm{End}(\pi_0) \arrow[r] & \pi_0 L_{K(n)} \mathbb{P}(E) \arrow[r] & \pi_0 L_{K(n)} \mathbb{P}(E \oplus E)
\end{tikzcd}$$

iso b/c (Fin) free

e.g. $n=1$, $E = E(\mathbb{F}_p, \hat{\mathbb{G}}_m)$

$\Rightarrow \mathrm{Alg}_{\mathbb{T}} = \delta$-rings

$\delta : \pi_0 \to \pi_0$ deg 0 (non-additive) power op

$\psi(x) = x^p + p\,\delta(x)$ $\quad p^{th}$ Adams operation (ring homo lifting Frobenius)

Γ is gen. by ψ under addition & composition

$\Rightarrow \mathrm{End}(\pi_0) = \hat{\Gamma}_{\mathfrak{m}} = \mathbb{Z}_p[\psi]$

(4) Dual Perspectives

Strickland prop 3.7

$$\mathbb{T}(E_0) = \mathbb{T}(E_0)_0 = \bigoplus_{m \geq 0} E_0^{\wedge} B\Sigma_m \simeq \bigoplus_{m \geq 0} E^0 B\Sigma_m$$

$$\mathbb{T}(E_0)^{\wedge}_{\mathfrak{m}} \simeq \pi_0 L_{K(n)} \mathbb{P}(E)$$

$\Rightarrow \mathbb{T}(E_0)$ acts on $\pi_0 R$ $\quad \forall\ R \in \mathrm{Alg}_{E, K(n)}$

$$\mathbb{T}(E_0) \underset{E_0}{\otimes} \pi_0 R \longrightarrow \pi_0 R$$

Dually, $\mathbb{T}(E_0)$ coacts on $\pi_0 R$ as follows:

$E^0 B\Sigma_m$ fin. free E_0-mod

$$\begin{tikzcd}
R^0 = \pi_0 R \arrow[r] \arrow[d, leftrightsquigarrow] & \pi_0 R \underset{E_0}{\otimes} E^0 B\Sigma_m \simeq R^0(B\Sigma_m) \arrow[d, leftrightsquigarrow] \\
E \to R \text{ in } \mathrm{Mod}_E \arrow[r, mapsto] & E \otimes B\Sigma_m \simeq \mathbb{P}_m E \to \mathbb{P}_m R \to R \text{ in } \mathrm{Mod}_E
\end{tikzcd}$$

Def $P_m : R^0 = \pi_0 R \longrightarrow \pi_0 R \otimes_{E_0} E^0 B\Sigma_m \simeq R^0(B\Sigma_m)$ by

$P_m(xy) = P_m(x)\, P_m(y)$

$$E \overset{x}{\underset{y}{\rightrightarrows}} X \quad \longmapsto \quad E \otimes B\Sigma_m \simeq \mathbb{P}_m E \overset{\mathbb{P}_m x}{\underset{\mathbb{P}_m y}{\rightrightarrows}} \mathbb{P}_m R \longrightarrow R$$

$$E \simeq E \otimes_E E \xrightarrow{xy} X \otimes_E X \quad \longmapsto \quad E \otimes B\Sigma_m \simeq \mathbb{P}_m (E \otimes_E E) \xrightarrow[\mathbb{P}_m(xy)]{} \mathbb{P}_m R \longrightarrow R$$

$$\overset{E}{\sim} \quad E \otimes_E E \otimes (B\Sigma_m \otimes B\Sigma_m) \simeq (E \otimes B\Sigma_m) \otimes_E (E \otimes B\Sigma_m) \xrightarrow{\mathbb{P}_m(x)\, \mathbb{P}_m(y)} \mathbb{P}_m R \otimes_E \mathbb{P}_m R \longrightarrow \mathbb{P}_m R \longrightarrow R$$

$\uparrow \mathrm{id}, \Delta$

$E \otimes B\Sigma_m$ ∃ diagonal Δ b/c $B\Sigma_m$ is suspension sp

$\Rightarrow P_m(xy) = P_m(x)\, P_m(y)$

Let $J_m = \bigoplus_{0<i<m} \mathrm{im}\left(E^0 B(\Sigma_i \times \Sigma_{m-i}) \xrightarrow{\text{transfer}} E^0 B\Sigma_m \right) \underset{\text{ideal}}{\subseteq} E^0 B\Sigma_m$

Then

$\overline{P}_m : \pi_0 R \longrightarrow R^0 B\Sigma_m \longrightarrow R^0 B\Sigma_m / J_m$ ring homo.

additive + $\overline{P}_m(xy) = \overline{P}_m(x)\, \overline{P}_m(y)$

$E^0 B\Sigma_m \xrightarrow{i^*} E^0(*)$

$B\Sigma_m \xleftarrow{i} *$

$R^0(X) \xrightarrow{P_m} R^0(X \times B\Sigma_m) \xrightarrow{i^*} R^0(X)$ lift of Frobenius

$x \longmapsto x^m$

$E \to X$ $\quad E \to E \otimes B\Sigma_m \simeq E_{h\Sigma_m} \longrightarrow R^{\otimes_E m}_{h\Sigma_m} \longrightarrow R$

action		coaction
$\pi_0 R \otimes_{E_0} \Gamma(E_0) \longrightarrow \pi_0 R$		$\pi_0 R \longrightarrow \pi_0 R \otimes_{E_0} \Pi(E_0)$
Γ additive operations	v.s.	$\overline{P}_m$ ring homo
= ker of transfer		= mod out by transfer