$K(1)$-LOCAL E_∞ RINGS
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ABSTRACT. In this talk, we will show that the free $K(1)$-local E_∞ ring on a class in degree zero has a particularly nice form. Namely, it is a free θ-algebra on a single generator. Understanding this brings us closer to the goal of finding "enough points" to show that Nullstellensatzian rings are interesting.

CONTENTS

1. INTRODUCTION

The main goal of this talk is to completely describe the structure of the free $K(1)$-local E_∞ ring on the sphere spectrum. Through several steps, we will show a $K(1)$-local equivalence between this free $K(1)$-local E_∞ ring and a sum of $K(1)$-local spheres, as well as recognize two power operations that provide the structure of a θ-algebra on this ring.

For comparison, consider the (not localized) free E_∞ ring on a spectrum X. This is given by

$$\mathrm{Free}_{E_\infty}(X) = \mathbb{S} \oplus X \oplus X^{\otimes 2}_{h\Sigma_2} \oplus X^{\otimes 3}_{h\Sigma_3} \oplus \cdots$$

When we take $X = \mathbb{S}$ the sphere spectrum, we get

$$\mathrm{Free}_{E_\infty}(X) = \mathbb{S} \oplus \mathbb{S} \oplus \mathbb{S}^{\otimes 2}_{h\Sigma_2} \oplus \mathbb{S}^{\otimes 3}_{h\Sigma_3} \oplus \cdots$$

This is a complicated ring, with many power operations. Each individual summand is also difficult to study- for example, $\mathbb{S}^{\otimes 2}_{h\Sigma_2} \simeq \Sigma^\infty_+ \mathbb{R}P^\infty$, an object with many cells and complicated homotopy groups.

On the other hand, if we work rationally and take the free rational E_∞ ring on the Eilenberg-MacLane spectrum $H\mathbb{Q}$ (for which we will use the shorthand $\mathbb{Q}$) then the output is simpler:

$$\mathrm{Free}^{\mathbb{Q}}_{E_\infty}(\mathbb{Q}) = \mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{Q}^{\otimes 2}_{h\Sigma_2} \oplus \mathbb{Q}^{\otimes 3}_{h\Sigma_3} \oplus \cdots \simeq \mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{Q} \oplus \cdots \simeq \mathbb{Q}[x].$$

In this ring, the only operations we have are addition and multiplication, and we completely understand its structure: it is a polynomial ring on a single generator, which we name x. Let's write down explicitly which summand each monomial corresponds to:

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$$\begin{array}{ccccccc} \mathbb{Q} & \oplus & \mathbb{Q} & \oplus & \mathbb{Q} & \oplus & \cdots \\ x & & x^2 & & x^3 & & \cdots \end{array}$$

The simplicity of this ring is a height zero phenomenon. Working $K(1)$-locally, so at height 1, the result is a bit more complicated. A result of McClure, which is Theorem 5 in [Hop], is the following.

Theorem 1.1. *Let E be a $K(1)$-local E_∞ ring spectrum. Then*

$$\mathrm{Free}^{K(1)}_{E_\infty E}(\mathbb{S}) \to E_*\{x, \theta x, \theta^2 x, \cdots\}$$

is an isomorphism of θ-algebras.

A θ-algebra is an algebra with operations θ and ψ such that $\psi(x) = x^p + p\theta(x)$ and such that ψ is a ring homomorphism. We write θx for $\theta(x)$. The requirement that ψ is a ring homomorphism also places restrictions on the values of θ on sums and products of elements, so that each operation is completely determined.

Remark 1.2. *In* [Hop]*, θ-algebras are called Frobenius algebras. In some sense, they are a mod p analog of lambda rings.*

We will focus on the case where E is the $K(1)$-local sphere.

Corollary 1.3. *Let $\mathbb{S}_{K(1)}$ be the $K(1)$-local sphere spectrum. Then*

$$\mathrm{Free}^{K(1)}_{E_\infty}(\mathbb{S}_{K(1)}) \to \mathbb{S}_{K(1)_*}\{x, \theta x, \theta^2 x, \cdots\}$$

is an isomorphism of θ-algebras.

Note that from this result we may recover the general case by smashing with E.

The right hand side is the free θ-algebra on a single generator, x. Explaining and sketching a proof of this theorem will be the main goal of this talk, but first we will explain why this result is relevant to the seminar.

2. Relevance to the Chromatic Nullstellensatz

Let's fix a prime p and a formal group law of height n over $\mathbb{F}_p$. Recall from Tomer's talk that there is a functor

$$\mathsf{Perf}_{\mathbb{F}_p} \to \mathsf{CAlg}(\mathsf{Sp}_{T(n)})$$

from perfect $\mathbb{F}_p$-algebras to $T(n)$-local spectra taking $F \mapsto E(F)$, where $E(F)$ is the Lubin-Tate spectrum .

A main theorem in [BSY22] is the following characterization of Nullstellensatzian rings.

Theorem 2.1. *A spectrum $R \in \mathsf{CAlg}(\mathsf{Sp}_{T(n)})$ is Nullstellensatzian $\iff$ there exists an algebraically closed field F of characteristic p such that $R \simeq E(F)$.*

For this to be useful, we wanted a notion of "having enough points;" i.e. having enough E_∞ ring maps $S \to E(F)$. This would show that Nullstellensatzian rings can capture interesting information.

Thinking optimistically, if we did have such ring maps

$$S \to E(F),$$

then by taking π_0, we'd get a map of θ-algebras

$$\pi_0 S \to \pi_0 E(F).$$

Now recall that if F is a perfect $\mathbb{F}_p$ algebra, then $\pi_0 E(F) \cong W(F)[[v_1, ..., v_{n-1}]]$, where $W(F)$ are the Witt vectors of F. We rely on the following fact.

Fact 2.2. *The assignment $F \mapsto W(F)$ defines a functor*

$$\mathsf{CRing} \to \Lambda\text{-}\mathsf{Ring}.$$

This functor is the right adjoint to the forgetful functor $\mathrm{fgt}\colon \Lambda\text{-}\mathsf{Ring} \to \mathsf{CRing}$.

In other words, the Witt vectors of F form a cofree lambda ring. See, e.g., [Haz08]. The lambda ring structure on $W(R)$ arises from the Frobenius automorphism on R, which by functoriality defines a Frobenius automorphism on $W(R)$. In any case, the adjunction provides a natural isomorphism

$$\mathrm{Hom}_{\Lambda\text{-}\mathsf{Ring}}(A, W(F)) \cong \mathrm{Hom}_{\mathsf{CRing}}(A, F),$$

allowing us to identify the map on π_0 with a map

$$\pi_0 S \to F.$$

But now, in the category of commutative rings, it is easy to produce maps into a ring F.

This argument doesn't yet show that we "have enough points," but it provides some corroborating evidence, and similar ideas will lead to a proof later in the seminar.

3. Proof of Theorem

We start by expanding out the LHS of the theorem statement. We have, as in the general case,

$$\mathrm{Free}_{E_\infty}^{K(1)}(\mathbb{S}_{K(1)}) \simeq \mathbb{S}_{K(1)} \oplus (\mathbb{S}_{K(1)})^{\otimes 2}_{h\Sigma_2} \oplus \cdots \oplus (\mathbb{S}_{K(1)})^{\otimes p-1}_{h\Sigma_{p-1}} \oplus (\mathbb{S}_{K(1)})^{\otimes p}_{h\Sigma_p} \oplus \cdots$$

However, $K(1)$-locally, we will see that the first $p-1$ terms are each equivalent to $\mathbb{S}_{K(1)}$, while $(\mathbb{S}_{K(1)})^{\otimes p}_{h\Sigma_p} \simeq \mathbb{S}_{K(1)} \oplus \mathbb{S}_{K(1)}$, and the remaining terms also split into sums of spheres. That is, we can rewrite the above as

$$\mathrm{Free}_{E_\infty}^{K(1)}(\mathbb{S}_{K(1)}) \simeq \mathbb{S}_{K(1)} \oplus \mathbb{S}_{K(1)} \oplus \cdots \oplus \mathbb{S}_{K(1)} \oplus (\mathbb{S}_{K(1)} \oplus \mathbb{S}_{K(1)}) \oplus \cdots$$

We'll start by studying the terms $(\mathbb{S}_{K(1)})^{\otimes n}_{h\Sigma_n}$ with $n < p$.

3.1. Summands with $n < p$.

Lemma 3.1. *When $n < p$, there is a $K(1)$-local equivalence $(\mathbb{S}_{K(1)})^{\otimes n}_{h\Sigma_n} \simeq \mathbb{S}_{K(1)}$.*

Proof sketch. Note that when $n < p$, the order of the symmetric group, $|\Sigma_n| = n!$, is coprime to p. In general, if a finite group G acts on X, then we can compose the norm map with the inclusion of the fixed points into X, as well as precompose with the quotient map to the orbits:

$$X_{hG} \xrightarrow{\mathrm{Nm}} X^{hG} \xrightarrow{i} X \xrightarrow{q} X_{hG}$$

When $|G|$ is invertible, the composition $X \to X$ is invertible since it corresponds to a multiplication by $|G|$. So, each of the compositions $X_{hG} \to X_{hG}$ and $X^{hG} \to X^{hG}$ is also invertible. Therefore, we can realize X_{hG} and X^{hG} as retracts of X. Then $\pi_*(X_{hG})$, for example, is the image of the idempotent witnessing the retraction applied to $\pi_* X$, which is $\pi_*(X)_{hG}$.

Using this, we can identify $\pi_*(\mathbb{S}_{K(1)})^{\otimes n}_{h\Sigma_n}$ with $(\pi_* \mathbb{S}^{\otimes n}_{K(1)})_{h\Sigma_n}$. But now, the Σ_n action is trivial, since it acts by permuting copies of the unit object in our symmetric monoidal category, so we may simplify further to $\pi_* \mathbb{S}^{\otimes n}_{K(1)} \simeq \pi_* \mathbb{S}_{K(1)}$. This equivalence on π_* implies a $K(1)$-local equivalence, so we see that $K(1)$-locally, $(\mathbb{S}_{K(1)})^{\otimes n}_{h\Sigma_n} \simeq \mathbb{S}_{K(1)}$.

□

3.2. pth Summand.

When $n = p$, the argument is more involved. We will use the notation $B\Sigma_{n+}$ for $L_{K(1)}\Sigma^\infty_+ B\Sigma_n$, the $K(1)$-localization of the suspension spectrum of the classifying space of the symmetric group. Recall that $(\mathbb{S}_{K(1)})_{h\Sigma_n} \simeq B\Sigma_{n+}$. We want to show the following result, called Lemma 3 in [Hop]:

Lemma 3.2. *Let ε be the map induced by $B\Sigma_p \to \mathrm{pt}$ and let tr be the transfer map. Then*

$$B\Sigma_{p_+} \xrightarrow{\varepsilon, \mathrm{tr}} \mathbb{S}_{K(1)} \oplus \mathbb{S}_{K(1)}$$

is a $K(1)$-local equivalence.

Remark 3.3. *A bit more generally, whenever we are working in a p-complete, 1-semiadditive category,* $\mathbb{1} \oplus \mathbb{1}$ *is a retract of* $B\Sigma_p \oplus \mathbb{1}$*. So, for instance,* $K(n)$*-locally,* $\mathbb{S}_{K(n)} \oplus \mathbb{S}_{K(n)}$ *is a retract of* $L_{K(n)}\Sigma^\infty_+ B\Sigma_p$*. However, there is only an* equivalence *at height 1.*

Recall from last time that for $K(1)$-local spectra, we have Tate vanishing; i.e. the norm map $X_{\Sigma_p} \xrightarrow{\simeq} X^{hG}$ is an equivalence. This implies that $K(1)$-locally,

$$B{\Sigma_p}_+ \simeq \mathbb{S}_{h\Sigma_p} \simeq \mathbb{S}^{h\Sigma_p}.$$

Smashing both sides with $K(1)$, we see that

$$K(1) \otimes \mathbb{S}_{h\Sigma_p} \simeq K(1) \otimes \mathbb{S}^{h\Sigma_p},$$

so after we take π_*, we see that

$$K(1)_*(B{\Sigma_p}_+) \simeq K(1)^*(B{\Sigma_p}_+).$$

Therefore, to compute the $K(1)$-homology, it suffices to compute the $K(1)$-cohomology, for which we can use an Atiyah-Hirzebruch spectral sequence:

$$H^*(B\Sigma_p; \pi_* K(1)) \implies K(1)^* B\Sigma_p.$$

To compute the input, we'll start with a similar trick to the one used in the previous section. Note that $C_p \hookrightarrow \Sigma_p$ is the inclusion of an index-prime-to-p subgroup. (Specifically, the index is $(p-1)!$.) On classifying spaces,

$$B{C_p}_+ \underset{\xleftarrow{\ \mathrm{tr}\ }}{\xrightarrow{\ i\ }} B{\Sigma_p}_+$$

the composition $i \circ \mathrm{tr}$ should be multiplication by the index, which is a unit, implying that $B{\Sigma_p}_+$ is a retract of $B{C_p}_+$. So, we will compute $H^*(BC_p; \pi_* K(1))$, which is a bit easier, and then identify two summands corresponding to $H^*(B\Sigma_p; \pi_* K(1))$.

To compute this, we can use a general fact for complex oriented cohomology theories.

Claim 3.4. *Let R be a complex oriented cohomology theory. Then,*

$$R^*(BC_p) \cong R[[t]]/[p](t),$$

where $[p](t)$ is the p-series for the formal group law associated to R.

Proof sketch. It follows from results in Tristan's talk that for R complex oriented, $H^*(BS^1; R) \cong R^*[[t]]$. To get from BS^1 to BC_p, we may start with the fibration

$$BC_p \to BS^1 \xrightarrow{\times p} BS^1.$$

Continuing to the left, there is also a fibration

$$S^1 \to BC_p \to BS^1.$$

The Atiyah-Hirzebruch spectral sequence associated to this fibration has the following signature:

$$H^*(BS^1; R^*(S^1)) \implies R^*(BC_p).$$

The cohomology of S^1 is an exterior algebra on some generator, say ε, of degree 1, so combining this with the result that the cohomology of BS^1 is a power series on one generator, we see that the input to the spectral sequence looks like $\Lambda^*(\varepsilon)[[t]]$. A sketch of the E_2 page is as follows:

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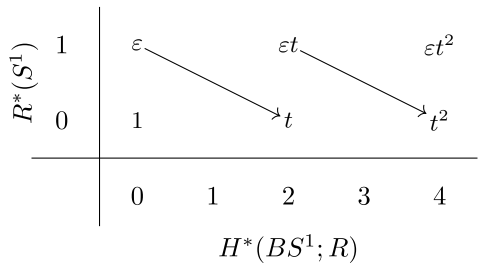

For degree reasons, the only possible differentials are d_2's. These are all determined, due to multiplicativity and the Leibniz rule, by the differential emanating from (0,1). Write d for this differential.

Claim 3.5. $\operatorname{im} d \cong [p](t)$

Some idea for why this is true is that the $[p]$-series must die by exactness- the next map in the sequence if $BS^1 \xrightarrow{\times p} BS^1$. To make this rigorous, we would need to argue that the $[p]$-series is indecomposable, for one thing. But, admitting this claim, we see that the spectral sequence collapses on the E_3 page, and we have shown the desired statement (at least at the level of the associated graded). □

Next, we apply this general fact to $R = K(1)$. Recall that for KU, the formal group law is the multiplicative formal group law $\mathbb{G}_m$, which can be written

$$x +_{\mathbb{G}_m} y = (x+1)(y+1) - 1$$

(neglecting the Bott class). So, the $[p]$-series for KU is

$$[p]_{\mathbb{G}_m}(t) = (t+1)^p - 1 = t^p + p(\cdots).$$

Since $K(1) \simeq \widehat{KU}^{Ad}/p$, the p-completed Adams summand of KU mod p, we have that the $[p]$-series for $K(1)$ is given by

$$[p](t) = t^p.$$

Therefore,

$$K(1)^*(BC_p) \cong K(1)^*[[t]]/t^p.$$

Note that as a module over $K(1)^*(\mathrm{pt})$ this is $K(1)^*\{1, \ldots, t^{p-1}\}$. In particular, it's finitely generated. This is already much nicer than the non-$K(1)$-local result.

Finally, we need to extract the summands corresponding to $K(1)^*(B\Sigma_p)$.

Claim 3.6. *There is an isomorphism of $K(1)^*(\mathrm{pt})$ modules*

$$K(1)^*(B\Sigma_p) \cong K(1)^*\{1, t^{p-1}\}.$$

We will not really justify this claim, but instead appeal to the fact that $K(1)^*(B\Sigma_p)$ corresponds to the fixed points of the $\mathbb{F}_p^\times$ action on $K(1)^*(BC_p)$. The action of some generator $c \in \mathbb{F}_p^\times$ takes a class t to ct, so in addition to the trivial summand, the t^{p-1} summand is also fixed since $t^{p-1} \mapsto c^{p-1}t^{p-1} = t^{p-1}$. (This uses that $c \in \mathbb{F}_p^\times$ is order $p-1$.)

Finally, to connect back to the lemma statement, we note that the trivial summand 1 corresponds to the map ε, basically by definition of ε. To see that the summand t^{p-1} corresponds to the transfer, it would take some more work. For example, it follows from the fact that $\mathrm{tr}(1) = [p](t)/t$, but we won't justify this.

In any case, the computation $K(1)^*(B\Sigma_p) \cong K(1)^*\{1, t^{p-1}\}$ shows that there is a $K(1)$-local equivalence $B\Sigma_{p+} \simeq \mathbb{S}_{K(1)} \oplus \mathbb{S}_{K(1)}$.

3.3. **Summands with $n > p$.** These summands are a bit more complicated. For $n = p + 1$ until $n = 2p-1$, one can use an argument with the inclusion $\Sigma_p \hookrightarrow \Sigma_n$ index-prime-to-p index subgroup to see that there is a $K(1)$-local equivalence between these spectra. For $n = 2p$, one needs to examine a different subgroup, like $\Sigma_p \times \Sigma_p$. For $n = p^2$, the argument requires a more complicated choice of subgroup.

3.4. **Power Operations.** Above, we gave a partial argument for how $K(1)$-locally, the free E_∞ ring on a generator in degree 0 was equivalent to a sum of spheres. However, we haven't yet realized the θ-algebra structure.

Lemma 3.2 exhibits the ε and transfer maps as a dual basis for $B\Sigma_p$ as a $K(1)^*(\mathrm{pt})$ module. We can form another basis dual to this by defining operations $\theta, \psi\colon \mathbb{S} \to B\Sigma_{p+}$ such that

- $\mathrm{tr} \circ \theta$ is multiplication by $(p-1)!$,
- $\varepsilon \circ \theta$ is 0,
- $\mathrm{tr} \circ \psi$ is 0, and
- $\varepsilon \circ \psi$ is 1.

Then, define a map e as follows. Start with the inclusion of a point $\mathrm{pt} \hookrightarrow \Sigma_p$, then take the classifying spaces, $B\mathrm{pt} \hookrightarrow B\Sigma_p$, and finally apply Σ^∞_+ to get

$$e\colon \mathbb{S} \to B\Sigma_{p+}.$$

One can compute that $\varepsilon \circ e = 1$ from the definitions of these maps, and that $\mathrm{tr} \circ e = p!$ by using the double coset formula for the (trivial) subgroup $\{1\} \subset \Sigma_p$. Then, one can show by evaluating the basis elements tr and ε that

$$\psi = e + p\theta.$$

Let E be a $K(1)$-local E_∞ ring spectrum. Given an element $x \in \pi_0 E$, define a map $P(x)\colon B\Sigma_{p+} \to E$ by starting with the map $\mathbb{S} \to E$ representing x, using the free E_∞-forgetful adjunction, and restricting the resulting map to the summand $\mathbb{S}^{\otimes p}_{h\Sigma_p} \simeq B\Sigma_{p+}$. Next, define $e(x) = P(x) \circ e$, $\theta(x) = P(x) \circ \theta$, and $\psi(x) = P(x) \circ \psi$. One can show that $e(x) = x^p$; it follows that $\psi(x) - x^p = p\theta(x)$.

Fact 3.7. *The map ψ is a ring homomorphism.*

We won't prove this, but emphasize that it is a height 1 phenomenon. One can define operations θ and ψ on $K(n)$-local spectra, but for general n, ψ will not be a ring homomorphism, and thus the ring structure may not be completely determined by these two operations.

3.5. **Conclusion.** With the two power operations defined, we can now identify the summands in the free $K(1)$-local E_∞ ring on $\mathbb{S}$ with the elements in the free θ-algebra. We claim that the correspondence is as follows:

$\mathrm{Free}^{K(1)}_{E_\infty}(\mathbb{S}_{K(1)}) \simeq$

$$\begin{array}{ccccccccccccccc} \mathbb{S}_{K(1)} & \oplus & \cdots & \oplus & \mathbb{S}_{K(1)} & \oplus & (\mathbb{S}_{K(1)} \oplus \mathbb{S}_{K(1)}) & \oplus & \cdots & \oplus & (\mathbb{S}_{K(1)} \oplus \mathbb{S}_{K(1)} \oplus \mathbb{S}_{K(1)}) & \oplus & \cdots & \oplus & (\mathbb{S}_{K(1)} \oplus \cdots) \\ x & & \ldots & & x^{p-1} & & x^p, \theta x & & \ldots & & x^{2p}, x^p\theta x, \theta x \theta x & & \ldots & & \theta^2 x, \ldots \end{array}$$

Note that the generators appearing in the sum are exactly the generators one sees in the free θ-algebra on x. Understanding this structure will contribute toward the proof of the Nullstellensatz in future talks.

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