Juvitop Bott element

February 10, 2021

Here's the deal with the Bott element. First, remember that we have this spectrum ku, connective complex K-theory, whose zeroth space is $\mathrm{BU} \times \mathbb{Z}$. So we have, for a space X, that

$$\pi_0 \mathrm{Map}(\Sigma^\infty_+ X, \mathrm{ku}) = \pi_0 \mathrm{Map}(X, \mathrm{BU} \times \mathbb{Z})$$

is the group of virtual vector bundles on X. In particular, $\pi_2\mathrm{ku}$ is the group of virtual vector bundles on $S^2 = \mathbb{C}P^1$, and we have a favorite one, $\mathcal{O}(1)$, coming from the canonical line bundle. We usually shift it to have rank 0 and use $\beta := [\mathcal{O}(1)] - 1$. So we've got this diagram

$$\mathbb{C}P^1 \to \mathbb{C}P^\infty = \mathrm{B}S^1 = \mathrm{BU}(1) \to \mathrm{BU}.$$

defining the Bott element.

On the other hand, we can include a cyclic group C_q into S^1 as the qth roots of unity, i.e. the kernel of the map $S^1 \to S^1$ given by $z \to z^q$. So we have a map

$$\mathrm{B}C_q \to \mathrm{B}S^1$$

We also have a little cofiber sequence

$$S^1 \overset{(-)^q}{\to} S^1 \to S^1/q \to S^2.$$

Consider the diagram:

$$\begin{array}{ccc} S^1/q & \dashrightarrow & \mathrm{B}C_q \\ \downarrow & & \downarrow \\ S^2 & \longrightarrow & \mathrm{B}S^1 \\ & & \downarrow \\ & & \mathrm{B}S^1 \end{array}$$

The long composite $S^1/q \to \mathrm{B}S^1$ is null, $\mathrm{B}S^1$ represents $H^2(-,\mathbb{Z})$ and $H^2(S^1/q) = \mathbb{Z}/q$, so can fill in the dotted map. (Alternatively, you could use the fact that $\pi_1 \mathrm{B}S^1 = 0$ and use the cofiber sequence).

Taking suspension spectra, we then get the following diagram:

$$\begin{array}{ccccc} \Sigma\mathbb{S}/q & \longrightarrow & \Sigma^\infty_+ \mathrm{B}C_q & & \\ \downarrow & & \downarrow & & \\ \Sigma^2\mathbb{S} & \longrightarrow & \Sigma^\infty_+ \mathrm{B}S^1 & \longrightarrow & \mathrm{ku} \end{array}$$

Now, it turns out that $\mathbb{S}/q$ is *dualizable*, so that

$$\mathrm{Map}(\mathbb{S}/q, -) \simeq D(\mathbb{S}/q) \otimes (-),$$

for some spectrum $D(\mathbb{S}/q)$. Which spectrum? Well, we can work it out by dualizing the defining cofiber sequence:

$$\Sigma^{-1}\mathbb{S}/q \to \mathbb{S} \xrightarrow{\cdot q} \mathbb{S} \to \mathbb{S}/q$$

becomes

$$\Sigma D(\mathbb{S}/q) \leftarrow \mathbb{S} \leftarrow \mathbb{S} \leftarrow D(\mathbb{S}/q)$$

so that

$$D(\mathbb{S}/q) = \Sigma^{-1}\mathbb{S}/q.$$

In particular:

$$\pi_n(X/q) := \pi_n(X \otimes \mathbb{S}/q) = \pi_0 \mathrm{Map}(\Sigma^n \mathbb{S}, X \otimes \mathbb{S}/q) = \pi_0 \mathrm{Map}(\Sigma^{n-1}\mathbb{S}/q, X).$$

Revisiting our diagram, we have produced an element $\beta \in \pi_2((\Sigma^\infty_+ \mathrm{B}C_q)/q)$ which lifts the mod q reduction of β in $\pi_2\mathrm{ku}$.

If we knew that the mod q homotopy groups had a ring structure, we would then get a split injection

$$\mathbb{Z}/q[\beta] \to \pi_*((\Sigma^\infty_+ \mathrm{B}C_q)/q).$$

Now, $\Sigma^\infty_+ \mathrm{B}C_q$ is a ring (it's like the 'group ring' of the space $\mathrm{B}C_q$, which is a topological abelian group), but $\mathbb{S}/q$ actually can't be made into a highly structured ring. However, when $q \geqslant 3$, we can define a pairing

$$\mathbb{S}/q \otimes \mathbb{S}/q \to \mathbb{S}/q$$

which is unital; and that's enough to get a ring structure on the homotopy groups, which is enough to define all those powers β^n of the Bott element.