Étale Cohomology: A Crash Course

(with a word on algebraic K-theory)

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Motivation: singular cohomology

Let X be a scheme. If X is locally of finite type over $\mathbb{C}$, we can endow $X(\mathbb{C})$ with the Euclidean topology and consider $H^i_{\text{sing}}(X(\mathbb{C}); A)$ for any abelian group A. This is a useful invariant. Recall that it equals the sheaf cohomology $H^i(X(\mathbb{C}); \underline{A})$.

But what if X was locally of finite type over $\overline{\mathbb{F}}_p$? Or $X = \operatorname{Spec} \mathbb{Z}$? We no longer have the Euclidean topology, but X has an underlying topological space, so we can form sheaf cohomology, I guess...

Proposition

Let X be an irreducible topological space. Then $H^i(X; \underline{A}) = 0$ for all $i \geq 1$.

Proof.

By irreducibility, every pair of nonempty open subsets of X intersect. This implies that $\underline{A}$ is flasque and hence has trivial higher cohomology. □

The underlying topological space of X is often irreducible. Thus, for sheaf cohomology purposes, it has "too few" open subsets!

So we should add more "open subsets." But what exactly do we add? We take a scheme-theoretic version of local isomorphisms, motivated by the inverse function theorem:

Definition

Let $\pi : Y \to X$ be a morphism of schemes.

- Suppose $X = \operatorname{Spec} R$ is affine. We say π is *standard étale* if it is isomorphic to an open subscheme of $\operatorname{Spec}(R[t]/g)[\frac{1}{g'}]$, where g in $R[t]$ is a monic polynomial, and g' denotes its derivative.
- In general, we say π is *étale* if, for all x in X and y in $\pi^{-1}(x)$, there exists an affine neighborhood U of x and a neighborhood V of y with $\pi(V) \subseteq U$ such that $\pi : V \to U$ is standard étale.

Examples

- Let k be a ring, and choose an integer n invertible in k. Then the endomorphism of $\mathbb{G}_{m,k} := \operatorname{Spec} k[x, x^{-1}]$ given by $x \mapsto x^n$ is standard étale, as we can take $g = t^n - x$. (Imagine this for $k = \mathbb{C}$!)

Examples (continued)

- Let l/k be a finite separable extension of fields. Then $\operatorname{Spec} l \to \operatorname{Spec} k$ is standard étale, as we can take g to be the minimal polynomial of a generator of l over k. Conversely, one can show that every étale morphism to $\operatorname{Spec} k$ from a connected scheme is of this form.
- Taking $g = t$ shows that the identity is standard étale, so open embeddings are étale. In general, étale morphisms are open.

Étale morphisms are stable under composition and base change.

Recall that presheaves are just contravariant functors from the category of open subsets to (Set), and sheaves are just presheaves whose sections glue along *covers*. This abstract perspective leads us to make the following:

Definition

The *étale site* $X_{\text{ét}}$ is the category of étale schemes over X, where a cover of Y in $X_{\text{ét}}$ is given by families $\{U_i \to Y\}_i$ of étale morphisms such that $\coprod_i U_i \to Y$ is surjective.

This yields a notion of sheaves, exact sequences, sheaf cohomology, etc.

Proposition

Let $X = \operatorname{Spec} k$, where k is a field. Fix a separable closure $\overline{k}$ of k. Then the category of abelian sheaves on $X_{\text{ét}}$ is equivalent to the category of *continuous* (i.e. every element has open stabilizer) $\operatorname{Gal}(\overline{k}/k)$-modules.

Proof.

Let F be an abelian sheaf on $X_{\text{ét}}$. Set $M := \varinjlim_l F(l)$, where l runs over finite Galois extensions of k in $\overline{k}$. Now $\operatorname{Gal}(\overline{k}/k)$ acts via the $\operatorname{Gal}(l/k)$, and M is evidently continuous for this action.

Conversely, let M be a continuous $\operatorname{Gal}(\overline{k}/k)$-module. For any finite separable extension l of k in $\overline{k}$, set $F(l) := M^{\operatorname{Gal}(\overline{k}/l)}$. To see that F is a sheaf, it suffices to check that, for all finite Galois extensions n/l in $\overline{k}$,

$$0 \to F(l) \to F(n) \to F(n \otimes_l n) = F(\textstyle\prod_{\operatorname{Gal}(n/l)} n) = \prod_{\operatorname{Gal}(n/l)} F(n)$$

is exact. We see it's exact at the first term, and $F(n) \to \prod_{\operatorname{Gal}(n/l)} F(n)$ is the map $m \mapsto (m - \sigma(m))_\sigma$, whose kernel is $F(n)^{\operatorname{Gal}(n/l)} = F(l)$. □

Under this identification, the global sections functor $F \mapsto F(k)$ corresponds to the $\text{Gal}(\overline{k}/k)$-invariants functor $M \mapsto M^{\text{Gal}(\overline{k}/k)}$. Thus étale cohomology on $\operatorname{Spec} k$ equals Galois cohomology!

Return to an arbitrary scheme X. Here's a big source of étale sheaves:

Theorem (Grothendieck)

Let Z be a scheme. The presheaf $Y \mapsto \text{Hom}_{(Sch)}(Y, Z)$ on $X_{\text{ét}}$ is a sheaf.

Examples

- The abelian sheaf $\mathbb{G}_m$ given by $Y \mapsto \mathcal{O}_Y(Y)^\times$ is represented by $\mathbb{G}_{m,\mathbb{Z}}$,
- The abelian sheaf μ_n given by $Y \mapsto \{y \in \mathcal{O}_Y(Y)^\times \mid y^n = 1\}$ is represented by $\mu_{n,\mathbb{Z}} := \operatorname{Spec} \mathbb{Z}[x]/(x^n - 1)$,
- The constant abelian sheaf $\underline{A}$ is represented by $\coprod_{a \in A} \operatorname{Spec} \mathbb{Z}$.

Suppose that n is invertible in $\mathcal{O}_X(X)$ and that $\mathcal{O}_X(X)$ contains a primitive n-th root of unity ζ. The Chinese remainder theorem identifies $\mathbb{Z}[\zeta][\frac{1}{n}][x]/(x^n - 1) = \prod_{j \in \mathbb{Z}/n\mathbb{Z}} \mathbb{Z}[\zeta][\frac{1}{n}]/(x - \zeta^j) \simeq \prod_{j \in \mathbb{Z}/n\mathbb{Z}} \mathbb{Z}[\zeta][\frac{1}{n}]$. Taking Spec yields an isomorphism $\mu_n \simeq \underline{\mathbb{Z}/n\mathbb{Z}}$ of abelian sheaves on $X_{\text{ét}}$.

Proposition (Kummer)

Suppose that n is invertible in $\mathcal{O}_X(X)$. Then $1 \to \mu_n \to \mathbb{G}_m \xrightarrow{n} \mathbb{G}_m \to 1$ is an exact sequence of abelian sheaves on $X_{\text{ét}}$.

Proof.

It's evidently exact at the first and second terms. For the third term, let Y be in $X_{\text{ét}}$, and consider a in $\mathbb{G}_m(Y) = \mathcal{O}_Y(Y)^\times$. The *relative spectrum* $\underline{\mathrm{Spec}}_Y \mathcal{O}_Y[t]/(t^n - a)$ is an étale cover of Y (as the image of nt^{n-1} there is already invertible), and after passing to this étale cover, we tautologically get an n-th root of a! □

As usual, $H^1_{\text{ét}}(X; F)$ is in bijection with isomorphism classes of F-bundles on $X_{\text{ét}}$.

Proposition (Hilbert 90)

We have a natural isomorphism $H^1(X; \mathbb{G}_m) = H^1_{\text{ét}}(X; \mathbb{G}_m)$.

If X is the spectrum of a field k, this implies that $H^1(\mathrm{Gal}(\overline{k}/k), \overline{k}^\times) = 0$.

Since étale morphisms are stable under base change, morphisms of schemes $f : X_1 \to X_2$ induce functors f_* and f^* between their étale sheaves.

Proposition

Let X be a 1-dimensional regular integral scheme. Write $j : \eta \to X$ for the generic point inclusion, and for any closed point x in X, write $i_x : x \to X$ for the closed point inclusion. We have an exact sequence

$$1 \to \mathbb{G}_m \to j_*\mathbb{G}_m \xrightarrow{\text{Div}} \bigoplus_x i_{x,*}\underline{\mathbb{Z}} \to 0$$

of abelian sheaves on $X_{\text{ét}}$.

Theorem (Artin)

Let X be locally of finite type over $\mathbb{C}$, and let A be a finite abelian group. Then we have a natural isomorphism $H^i_{\text{ét}}(X; \underline{A}) = H^i(X(\mathbb{C}); \underline{A})$.

Altogether, we see that étale cohomology is an invariant for schemes that combines both Galois cohomology and singular cohomology.

Let X be a connected smooth proper curve over an algebraically closed field k, and suppose n is invertible in k. When $k = \mathbb{C}$, we can compute $H^i_{\text{ét}}(X; \underline{\mathbb{Z}/n\mathbb{Z}})$ via singular cohomology. But for general k, we can still compute it *purely algebraically*, and it's what you expect!

Proposition

Write g for the genus of X. We have $H^0_{\text{ét}}(X; \underline{\mathbb{Z}/n\mathbb{Z}}) \cong \mathbb{Z}/n\mathbb{Z}$, $H^1_{\text{ét}}(X; \underline{\mathbb{Z}/n\mathbb{Z}}) \cong (\mathbb{Z}/n\mathbb{Z})^{2g}$, $H^2_{\text{ét}}(X; \underline{\mathbb{Z}/n\mathbb{Z}}) \cong \mathbb{Z}/n\mathbb{Z}$, and all other $H^i_{\text{ét}}(X; \underline{\mathbb{Z}/n\mathbb{Z}}) \cong 0$.

Proof.

Since X is connected, $H^0_{\text{ét}}(X; \underline{\mathbb{Z}/n\mathbb{Z}})$ is immediate. Next, since n is invertible in $\mathcal{O}_X(X)$ and $\mathcal{O}_X(X)$ contains a primitive n-th root of unity, we can replace $\underline{\mathbb{Z}/n\mathbb{Z}}$ with μ_n. The Kummer short exact sequence yields a long exact sequence

$$\cdots \to H^{i-1}_{\text{ét}}(X; \mathbb{G}_m) \to H^i_{\text{ét}}(X; \mu_n) \to H^i_{\text{ét}}(X; \mathbb{G}_m) \xrightarrow{n} H^i_{\text{ét}}(X; \mathbb{G}_m) \to \cdots .$$

We have $\mathcal{O}_X(X)^\times = k^\times$ because X is connected and proper.

Proof (continued).

As k is algebraically closed, we see that $k^\times \xrightarrow{n} k^\times$ is surjective. Thus $H^1_{\text{ét}}(X; \mu_n)$ equals the n-torsion in $H^1_{\text{ét}}(X; \mathbb{G}_m)$. Hilbert 90 implies that $H^1_{\text{ét}}(X; \mathbb{G}_m) = \operatorname{Pic} X$, and recall that we have a short exact sequence

$$0 \to \operatorname{Pic}^0 X \to \operatorname{Pic} X \xrightarrow{\deg} \mathbb{Z} \to 0.$$

Now $\operatorname{Pic}^0 X$ is the k-points of a g-dimensional abelian variety over k, and since n is invertible in k, its n-torsion is isomorphic to $(\mathbb{Z}/n\mathbb{Z})^{2g}$. (For the same reason, multiplication by n is surjective on $\operatorname{Pic}^0 X$.) As $\mathbb{Z}$ is torsion-free, we see that $H^1_{\text{ét}}(X; \mu_n) \cong (\mathbb{Z}/n\mathbb{Z})^{2g}$.

I claim that $H^i_{\text{ét}}(X; \mathbb{G}_m) = 0$ for all $i \geq 2$. This would conclude the proof, because then $H^2_{\text{ét}}(X; \mu_n)$ would equal $\operatorname{Pic} X / n \operatorname{Pic} X$, which is $\mathbb{Z}/n\mathbb{Z}$ by the degree short exact sequence. The Kummer long exact sequence would also show that the other $H^i_{\text{ét}}(X; \mu_n) = 0$.

To prove $H^i_{\text{ét}}(X; \mathbb{G}_m) = 0$ for all $i \geq 2$, we use the divisor short exact sequence. As you might expect, f_* is exact for closed embeddings f.

Proof (continued).

Furthermore, a short argument using the étale version of local rings (i.e. *strictly henselian* local rings), checking isomorphisms at stalks, classical Hilbert 90, and Tsen's theorem shows that $Rj_*\mathbb{G}_m = j_*\mathbb{G}_m$. Therefore the divisor short exact sequence and the Grothendieck–Leray spectral sequence yield a long exact sequence

$$\cdots \to \bigoplus_x H^{i-1}_{\text{ét}}(x; \underline{\mathbb{Z}}) \to H^i_{\text{ét}}(X; \mathbb{G}_m) \to H^i_{\text{ét}}(\eta; \mathbb{G}_m) \to \bigoplus_x H^i_{\text{ét}}(x; \underline{\mathbb{Z}}) \to \cdots .$$

As x is the spectrum of an algebraically closed field, its higher Galois and hence étale cohomology vanish. Finally, η is the spectrum of a field with transcendence degree 1 over an algebraically closed field, so Tsen's theorem says that its Galois cohomology vanishes in degree ≥ 2. This yields the aforementioned vanishing of $H^i_{\text{ét}}(X; \mathbb{G}_m)$ for $i \geq 2$. □

Many theorems in étale cohomology (proper base change, finitude of cohomology, Poincaré duality, etc) are bootstrapped from this calculation.

Thomason spectral sequence

Let q be an odd prime power. Let k be $\mathbb{Z}[\frac{1}{q}]$ or one of

$$\{\text{global field, ring of integers of local field, separably closed field}\}$$

where q is invertible.

Theorem (Thomason)

Let X be a regular separated scheme of finite type over k. Then there exists a spectral sequence with differentials d_r in bidegree $(-r, r-1)$ and

$$E_2^{s,t} = H_{\text{ét}}^{-s}(X; \mu_q^{\otimes(t/2)}) \Rightarrow K_{s+t}^{(\beta)}(X; \mathbb{Z}/q\mathbb{Z})$$

concentrated in degrees $s \leq 0$ and $2|t$, where $K_{s+t}^{(\beta)}(X; \mathbb{Z}/q\mathbb{Z})$ denotes Bott-inverted algebraic K-theory with coefficients in $\mathbb{Z}/q\mathbb{Z}$.

For example, if $H_{\text{ét}}^{i}(X; \mu_q^{\otimes(t/2)}) = 0$ for $i \geq 3$ (as we've seen for our curves, but also it holds for $\operatorname{Spec} \mathbb{Z}[\zeta_q][\frac{1}{q}]$ and $\operatorname{Spec} \mathbb{Z}[\frac{1}{q}]$), this spectral sequence immediately degenerates and is super simple.

Closing remarks

When X is an open subscheme of the spectrum of the ring of integers of a number field, the divisor short exact sequence applies. And if n is invertible on X, the Kummer short exact sequence applies. One can use these, along with Galois cohomology computations from class field theory, to calculate étale cohomology groups of X.

References:

- L. Fu, *Etale Cohomology Theory*.
- R. Thomason, "Algebraic K-theory and etale cohomology."
- J. Milne, *Arithmetic Duality Theorems*.

Thank you!