Hodge Map & Witt groups.

Goal of §3: splitting of symplectic K-theory of $\mathbb{Z}$.

algebraic → ordinary alg. K-theory of $\mathbb{Z}$

topological → Witt groups

Theorem 3.5 (FGV). For $q=p^n$ odd (p prime):

$$KSp_i(\mathbb{Z};\mathbb{Z}/q) \xrightarrow{\sim} \left(K_i(\mathbb{Z};\mathbb{Z}/q)\right)^{(+)} \oplus \left(\pi_i(ku;\mathbb{Z}/q)\right)^{(-)}$$

$$\overset{k\geq 1}{\Rightarrow} \quad KSp_{4k-2}(\mathbb{Z};\mathbb{Z}/q) \cong H^2(\mathrm{Spec}(\mathbb{Z}');\mu_q^{\otimes 2k}) \oplus \mathbb{Z}/q$$

$\mathbb{Z}' = \mathbb{Z}[1/p]$

Relevance for paper?

- describe symplectic K-theory in terms of familiar things

  i.e. ordinary alg. K-theory + cmplx top. K-theory.

- main theorem (3.5 in paper) ⇝ used to prove surjectivity of "CM classes $\to KSp$".

  I.e. that CM classes exhaust symplectic K-theory – using explicit description of $KSp_{4k-2}(\mathbb{Z};\mathbb{Z}/q)$

Outline

- define Hodge & Betti maps

$KSp(\mathbb{Z}) \to ku$ in $ho(Sp)$ — $KSp(\mathbb{Z}) \to K(\mathbb{Z})$ of spectra

- Look at result on $\pi_i(-;\mathbb{Z}/q)$ for $q = p^n$, p prime.
- project onto an Eigenspaces under ψ^{-1} (Adams op.)

- relate Hermitian & symplectic K-theory

$$KH(??) \longrightarrow KSp(\mathbb{Z})$$ $\simeq$ once 2 is inverted

- interpret Hodge & Betti on KH

key tool: (Hermitian K-theory w/ 2 inverted)

$$\| $$

$$\left(\tfrac{1}{2}\text{-Alg. K theory}\right)^{(+)} \oplus \left(\tfrac{1}{2}\text{-Witt grp}\right)$$

show this decomposition corresponds to $(\text{Betti})^{(+)} \oplus (\text{Hodge})^{(-)}$

under identification of $KH[\tfrac{1}{2}]$ and $KSp[\tfrac{1}{2}]$.

Recollections – definition of KSp.

General procedure for constructing $KSp(R)$

$KSp(\mathbb{Z})$ def^n for this talk:

$SP(\mathbb{Z}) :=$ groupoid

objects: pairs (L, b)

- L f.g. free $\mathbb{Z}$-module
- $b : L \times L \to \mathbb{Z}$ skew-symm. pairing
- $\mathrm{adjoint}(b) : L \xrightarrow{\sim} L^\vee$.

morphisms: $(L,b) \longrightarrow (L',b')$ given by $\mathbb{Z}$-linear maps

- $f : L \xrightarrow{\sim} L'$
- $b'(fx, fy) = b(x,y) \quad \forall x, y$.

symm. mon. structure on $SP(\mathbb{Z})$:

$$\langle (L,b), (L',b') \rangle \longmapsto (L,b) \oplus (L',b') \overset{\shortparallel}{} (L \oplus L', b \oplus b')$$

- Group complete WRT $\oplus$ to get a spectrum $KSp(\mathbb{Z})$ (∞-loop space)

& map

$$\underset{\text{monoid}}{|SP(\mathbb{Z})|} \xrightarrow[\text{map}]{\text{Group completion}} \underset{\text{group}}{\Omega^\infty KSp(\mathbb{Z})}$$

(Really: E_∞ monoids/ groups)

- $KSp_i(\mathbb{Z};\mathbb{Z}/q) := \pi_i(KSp(\mathbb{Z});\mathbb{Z}/q)$

$$= \pi_0 \mathrm{Maps}(\underbrace{\mathbb{S}^0/q}_{\text{cofiber of mult. by } q}, KSp(\mathbb{Z}))$$

- $KSp_i(\mathbb{Z};\mathbb{Z}_p) = \varprojlim_n KSp_i(\mathbb{Z};\mathbb{Z}/p^n)$

[Can make these definitions for any R, but we don't need to.]

The Betti map

- construct $K(\mathbb{Z})$ by group completing $P(\mathbb{Z})$

the monoid (in gpds) $P(\mathbb{Z})$ of free $\mathbb{Z}$-modules & module $\simeq$ w/ monoidal structure given by direct sum.

"forget":
$$SP(\mathbb{Z}) \longrightarrow P(\mathbb{Z}) \quad \text{(respects mon. structure)}$$
$$(L,b) \longmapsto L$$

functoriality

$$c_B : KSp(\mathbb{Z}) \longrightarrow K(\mathbb{Z})$$

$\mathbb{Z}/q$ coeffs defined in same way

$$\rightsquigarrow \quad c_B = KSp_i(\mathbb{Z};\mathbb{Z}/q) \longrightarrow K_i(\mathbb{Z};\mathbb{Z}/q)$$

The Hodge map

Diagram of groupoids:

$$\begin{tikzcd} SP(\mathbb{Z}) \arrow[r, "(-)\otimes\mathbb{R}"'] & SP(\mathbb{R}^{top}) & \mathcal{U}(\mathbb{C}^{top}) \arrow[l, "(L,h)\mapsto(L_{\mathbb{R}},\, \mathrm{Im}\, h)"'] \end{tikzcd} \qquad (+)$$

- Top-enriched* groupoid of symplectic $\mathbb{R}$-modules
- $\mathbb{R}$ has topology $\Rightarrow$ morphisms have topology
- top-enriched geom. realization

- Top-enriched groupoid of $\mathbb{C}$-vector spaces & Hermitian forms
- top-enriched geom. realization

all groupoids have monoidal structure $\oplus$ compatible w/ maps.

$\therefore$ apply B^∞ $(+)$ $\rightsquigarrow$

$$KSp(\mathbb{Z}) \longrightarrow KSp(\mathbb{R}^{top}) \longleftarrow ku$$

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* $SP(\mathbb{Z})$ = discrete $\mathbb{E}_\infty$-monoid

$SP(\mathbb{R}^{top})$, $\mathcal{U}(\mathbb{C}^{top})$ non-discrete.

(I.e. we were secretly working w/ groupoids in a Top enriched setting to begin with.)

Terminology: group completion of $\mathcal{U}(\mathbb{C}^{top}) \rightsquigarrow ku$.

connective complex K-theory. ($\pi_* ku = 0$ for $*<0$)

connective cover of KU.

$$|\mathcal{P}(\mathbb{R}^{top})| \cong \coprod_g BSp_{2g}(\mathbb{R}) \qquad |\mathcal{U}(\mathbb{C}^{top})| \cong \coprod_g BU(g)$$

$\cong$ of top. monoids

$$\rightsquigarrow \begin{array}{ccc} |KSp(\mathbb{R}^{top})| & \xleftarrow{\sim} & |\mathcal{U}(\mathbb{C}^{top})| \\ \downarrow & & \downarrow \\ \Omega^\infty KSp(\mathbb{R}^{top}) & \underset{(*)}{\xleftarrow{\sim}} & \Omega^\infty ku \end{array}$$

b/c Group completion is a functor.

$$\therefore \text{ get } KSp(\mathbb{Z}) \longrightarrow KSp(\mathbb{R}^{top}) \longrightarrow ku \quad \text{in } ho(Sp).$$

$c_H :=$

Enough to talk about $\pi_* c_H$.

> The **Hodge map** $c_H: KSp_i(\mathbb{Z}; \mathbb{Z}/q) \longrightarrow \pi_i(ku; \mathbb{Z}/q)$ is the induced map on homotopy (mod q).

Rmk. Betti / Hodge terminology from connection w/ $\mathcal{A}_g$.

roughly: $BSp_{2g}(\mathbb{Z}) \longleftrightarrow \mathcal{A}_g$

- Hodge comes from chern classes of Hodge bundle / $\mathcal{A}_g$.
- Betti map comes from Betti realization

The Adams action on K-theory and ku.

Both algebraic K-theory and ku-theory carry an involution ψ^{-1}. [Adams operation.]

Alg. K-theory case: recall from Lucy's talk, $\mathrm{Proj}(R)^{\simeq}$ has ψ^{-1} given by $P \longmapsto \mathrm{Hom}(P, R)$

$\rightsquigarrow$ involution ψ^{-1} on htpy

ku case: ψ^{-1} ~very~ explicit. $\psi^{-1} = (-1)^k$ on $\pi_{2k} ku$.

$\therefore$ on $\pi_{2k}(ku; \mathbb{Z}/q)$.

Given a free module $/R$ w/ an involution ι:

If 2 is invertible in R, then:

$$V = \underbrace{V^{(+)}}_{\{v:\ \iota v = v\}} \oplus \underbrace{V^{(-)}}_{\{w \mid \iota w = -w\}}.$$

Clearly $V^{(+)} \oplus V^{(-)} \subseteq V$

$\supseteq$: $\forall x \in V$,

$x = \frac{1}{2}\left[(x - \iota x) + (x + \iota x)\right]$

Map of Thm (3.5)

$$KSp_i(\mathbb{Z}; \mathbb{Z}/q) \xrightarrow{c_B^{(+)} \oplus c_H^{(-)}} \left(K_i(\mathbb{Z}; \mathbb{Z}/q)\right)^{(+)} \oplus \left(\pi_i(ku; \mathbb{Z}/q)\right)^{(-)}$$

Outline: Proof of Main Theorem (3.5)

(I) relate $KSp(\mathbb{Z})$ to Hermitian K-theory

$$(\dagger)\quad KH(\mathbb{Z},-1)[\tfrac{1}{2}] \xrightarrow{\sim} KSp(\mathbb{Z})[\tfrac{1}{2}] \qquad \text{[Lemma 3.13]}$$

will remind.

(II) decompose $KH(A,\varepsilon)[\frac{1}{2}]$ using ordinary alg. K-theory & Witt groups

(apply to $K(\mathbb{Z},-1)[1/2]$)

$$(\star)\quad KH_i(A,\varepsilon)[\tfrac{1}{2}] = \left(K_i(A)[1/2]\right)^{(+)} \oplus W_i(A,\varepsilon)[1/2]$$

will define

(Karoubi result & strengthening to spectra)

don't really need here but it's nice.

(III) $(\dagger)$ & $(\star)$ $\rightsquigarrow$ map for $q=p^n$ odd:

$$\begin{array}{ccc}
\pi_i(KH(\mathbb{Z},-1);q) & \xrightarrow{\sim} & K_i(\mathbb{Z};\mathbb{Z}/q)^{(+)} \oplus W_i(\mathbb{Z};-1)/q \\
\downarrow s & & \downarrow \quad \text{show } \cong \\
KSp_i(\mathbb{Z};\mathbb{Z}/q) & \xrightarrow{c_B^{(+)} \oplus c_H^{(-)}} & K_i(\mathbb{Z};\mathbb{Z}/q)^{(+)} \oplus \pi_i(ku;\mathbb{Z}/q)^{(-)}
\end{array}$$

3.5 Proof, (I) Produce $KH(\mathbb{Z},-1) \longrightarrow KSp(\mathbb{Z})$.

Recollections: Hermitian K-theory (from Lucy's talk.)

Karoubi's approach:

Input: A & involution $A^{op} \longrightarrow A$, $a \mapsto \bar{a}$

central element ε

[e.g. $\mathbb{Z}$, trivial involution, $\varepsilon = 1$ or $\varepsilon = -1$.]

consider ε-symmetric sesquilinear forms on A.

collection of all $= Sesq(A)$, group under $\oplus$.

{ Proj. A-mod M, form $\varphi: M \times M \longrightarrow A$ s.t. $\varphi(xa, y) = \bar{a}\varphi(x,y)$, $\varphi(x, ya) = \varphi(x,y)a$. }

- (M, φ) is ε-symm. if
$$\varphi(x,y) = \varepsilon \overline{\varphi(y,x)}, \quad \forall x,y \in M.$$

- (M, φ) is perfect if $M \xrightarrow{\sim} M^t$ ← adjoint

where $M^t := \{ f: M \to A \text{ of } \mathbb{Z}\text{-mods} \mid f(\underset{\text{in } M}{x}\underset{\text{in } A}{a}) = \bar{a} f(x) \}$

$C_2 \circlearrowright Sesq(A)$ by $(M, \varphi) \longmapsto (M, \varepsilon\varphi^*)$ where $\varphi^*(x,y) := \overline{\varphi(y,x)}$

$$Sesq(A)_{C_2} \xrightarrow{\text{Norm}} Sesq(A)^{C_2}$$

$Q(A,\varepsilon)$: groupoid w/ these obs, monoid in grpds under $\oplus$

$\cup\!|$

{ sesq. linear forms whose norm is ε-perfect } = ε-quadratic forms

In the case $A = \mathbb{Z}$, $\varepsilon = -1$:

$Q(\mathbb{Z},-1)$ = groupoid of free ab. groups L w/

- unimodular (-1)-symmetric form $\omega: L \times L \to \mathbb{Z}$ (skew-symm)
- quadratic form $q: L \to \mathbb{Z}/2$ s.t.
  $q(x+y) - q(x) - q(y) = \omega(x,y) \pmod 2$

come from the requirement that $\omega + \varepsilon\omega^*$ be perfect.

"quadratic refinement of ω"

morphisms = compatible isoms of $\mathbb{Z}$-modules

monoidal structure from $\oplus$

The map [see 3.6.1]

Monoidal map of groupoids $Q(\mathbb{Z},-1) \longrightarrow SP(\mathbb{Z})$: $(L,\omega,q) \longmapsto (L,\omega)$

$\overset{B^\infty}{\rightsquigarrow}$ $KH(\mathbb{Z},-1) \longrightarrow KSp(\mathbb{Z})$

group completion: $|Q(\mathbb{Z},-1)| \to \Omega^\infty K(\mathbb{Z},-1)$.

Why $\simeq$ after inverting 2?

Essentially: data of q becomes trivial. [see Lemma 3.13]

$\rightsquigarrow$ get map on $\pi_i(-;\mathbb{Z}/q)$ which is an $\simeq$ for $q=p^n$ odd.

3.5 Proof, II

Map KH to K:

works for any (R, ε).

$$Q(\mathbb{Z},-1) \xrightarrow{\text{forget}} P(\mathbb{Z}) \xrightarrow[\text{hyp}]{\text{hyperbolize}} Q(\mathbb{Z},-1)$$

(M projective $\mathbb{Z}$-module)

$\mapsto$

$M \oplus M^t \simeq M \oplus M^\vee$ as a $\mathbb{Z}$-module

form: $\langle (x,f),(y,g)\rangle \longmapsto f(y)$.

$q(x,f) = f(x) \pmod 2$.

$$\begin{array}{ccccc} KH(\mathbb{Z},-1) & \xrightarrow{F} & K(\mathbb{Z}) & \xrightarrow{\text{hyp}} & KH(\mathbb{Z},-1) \\ \wr\| & & \wr\| & & \wr\| \\ K(\mathbb{Z},-1) & \xrightarrow{\psi^{-1}\cdot F} & K(\mathbb{Z}) & \xrightarrow{\text{hyp}\cdot\psi^{-1}} & KH(\mathbb{Z},-1) \end{array}$$

htpy comm.

$$\pi_i KH(\mathbb{Z},-1) \xrightarrow{F} \pi_i K(\mathbb{Z}) \xrightarrow{\text{hyp}} KH(\mathbb{Z},-1) \qquad (\Delta)$$

$(\pi_i K(\mathbb{Z}))^{(+)}$ $\qquad$ $(\pi_i K(\mathbb{Z}))_{\psi^{-1}}$. $\leftarrow$ ψ^{-1} coinvars.

ψ^{-1}-invariants

Def^n of Witt Groups

Higher Witt Groups $W_i(\mathbb{Z},-1) := \mathrm{Coker}\left(K_i(\mathbb{Z}) \xrightarrow{hyp} KH_i(\mathbb{Z},-1) \right).$

rmks. • In general, $KH_0(A,\varepsilon)$ = Grothendieck group of quad forms $/\sim$

= Grothendieck–Witt group of A.

check: $\mathrm{coker}\left(K_0(\mathbb{Z}) \longrightarrow KH_0(\mathbb{Z},-1)\right)$

matches your usual def^n.

$\{$quad forms / hyperbolic forms$\}$

• Higher GW-groups $:= \pi_i KH(-)$.

(A) $\rightsquigarrow$ Splitting for $i > 0$:

$$KH_i(\mathbb{Z},-1)[\tfrac{1}{2}] \longrightarrow K_i(\mathbb{Z})[\tfrac{1}{2}]^{(+)} \oplus W_i(\mathbb{Z},-1)[\tfrac{1}{2}]$$

3.5 Proof, III

"Betti maps "match up": ✓

By def.ⁿ of forgetful functors.

Assume $KSp_{4k-2}(\mathbb{Z};\mathbb{Z}/p) \twoheadrightarrow \pi_{4k-2}(ku;\mathbb{Z}/p) \simeq \mathbb{Z}/p.$

[see 5.2]

"The map" $W_i(\mathbb{Z},-1)/q \longrightarrow \pi_i(ku;\mathbb{Z}/q)^{(-)}$ is an isom.

- abstractly isom. groups.

$$\begin{array}{ccc} \pi_i(KH(\mathbb{Z},-1);q) & \xrightarrow{\sim} & K_i(\mathbb{Z};\mathbb{Z}/q)^{(+)} \oplus W_i(\mathbb{Z},-1)/q \\ \downarrow s & & \downarrow \\ KSp_i(\mathbb{Z};\mathbb{Z}/q) & \xrightarrow{c_B^{(+)} \oplus c_H^{(-)}} & K_i(\mathbb{Z};\mathbb{Z}/q)^{(+)} \oplus \pi_i(ku;\mathbb{Z}/q)^{(-)} \end{array}$$

⇝ • STP $KSp_i(\mathbb{Z};\mathbb{Z}/q) \longrightarrow K_i(\mathbb{Z};\mathbb{Z}/q)^{(+)} \oplus \pi_i(ku;\mathbb{Z}/q)^{(+)}$

is surjective.

Use Proposition 3.12!

Proof of second part of 3.5

Want: $KSp_{4k-2}(\mathbb{Z};\mathbb{Z}/q) \cong H^2(\mathrm{Spec}(\mathbb{Z}'); \mu_q^{\otimes 2k}) \oplus \mathbb{Z}/q$

Know: $KSp_i(\mathbb{Z};\mathbb{Z}/q) \xrightarrow{\sim} (K_i(\mathbb{Z};\mathbb{Z}/q))^{(+)} \oplus (\pi_i(ku;\mathbb{Z}/q))^{(-)}$

Identify $\pi_{4k-2}(ku;\mathbb{Z}/q)^{(-)}$:

- ψ^{-1} acts by $(-1)^i$ on $\pi_{2i} ku$

  $\rightsquigarrow$ entire group is (-1)-eigenspace.
- $\pi_{4k-2}(ku;\mathbb{Z}/q) = \mathbb{Z}/q$ (mod q Bott periodicity)

Identify $(K_i(\mathbb{Z};\mathbb{Z}/q))^{(+)}$: results on Bott inverted K-theory.

Compatible w/ ψ^{-1}.

(1) For $k \geq 1$, $K_{4k-2}(\mathbb{Z};\mathbb{Z}/q) \xrightarrow{\sim} K^{(\beta)}_{4k-2}(\mathbb{Z};\mathbb{Z}/q) \xrightarrow{\sim} K^{(\beta)}_{4k-2}(\mathbb{Z}';\mathbb{Z}/q)$

(2) $K^{(\beta)}_{4k-2}(\mathbb{Z}';\mathbb{Z}/q)^{(+)} \cong H^2(\mathrm{Spec}(\mathbb{Z}'); \mu_q^{\otimes k})$

(Lemma 2.14)

Thank you!