## Discussion section 5

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## Abstract

The following questions develop some ideas . Use them as jumping off points to explore ideas you find exciting or would like to understand better. They are in no particular order; feel free to jump around at your leisure.

Question 1. Recall that  $\pi_i(X; \mathbb{Z}/q) := \pi_0 \operatorname{Map}_{\operatorname{Sp}}(\mathbb{S}^{i-1}/q, X)$ , where  $\mathbb{S}^n/q$  is the cofiber of the multiplication by q map on  $\mathbb{S}^n$ . Show by example that

- i) Show by example that  $\pi_i(-;\mathbb{Z}/q) \neq \pi_i(-) \otimes_{\mathbb{Z}} \mathbb{Z}/q$ .
- ii) Show that  $\pi_i \operatorname{Map}_{\operatorname{Sp}}(-, \mathbb{S}^n/q \otimes H\mathbb{Z})$  computes  $H\mathbb{Z}/q^i$ .

**Question 2.** Read remark 3.4 of [FGV20] and understand the relationship between the Hodge/Betti maps as defined in section 3 and the classical Hodge bundle/Betti realization related to PPAVs.

Question 3. Deduce Corollary 3.6 from Theorem 3.5 [FGV20].

**Question 4.** Assuming  $\operatorname{KSp}_i(\mathbb{Z}; \mathbb{Z}/q) \xrightarrow{\sim} K_i(\mathbb{Z}; \mathbb{Z}/q)^+ \oplus \pi_i(ku; \mathbb{Z}/q)^{(-)}$ , deduce that  $\operatorname{KSp}_{4k-2}(\mathbb{Z}; \mathbb{Z}/q) \simeq H^2\left(\operatorname{Spec}\left(\mathbb{Z}\left[\frac{1}{q}\right]\right)\right); \mu_q^{\otimes 2k}) \otimes \mathbb{Z}/q.$ 

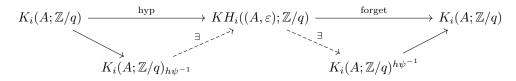
**Question 5.** Example 3.9 of [FGV20] claims that the definition of the Hermitian K-theory as  $\varepsilon$ -quadratic forms implies that  $KH(\mathbb{Z}, -1) = B^{\infty}(Q(\mathbb{Z}, -1))$  where  $Q(\mathbb{Z}, -1)$  is the groupoid consisting of

- objects: a triple  $(L, \omega, q)$  where L is a finitely-generated free  $\mathbb{Z}$ -module,  $\omega: L \times L \to \mathbb{Z}$  is a skew-symmetric form, and q is a quadratic refinement of  $\omega$ , i.e.  $q: L/2 \to \mathbb{Z}/2$  such that  $\omega(x, y) = q(x + y) - q(x) - q(y)$ .
- a morphism  $(L, \omega, q) \to (M, \eta, p)$  is an isomorphism of  $\mathbb{Z}$ -modules  $f : L \xrightarrow{\sim} M$  such that  $f^*(\eta) = \omega$ .

Work this out in detail from the definition of  $Q(A, \varepsilon)$ .

**Question 6.** Show that the hyperbolization of a projective *A*-module has perfect symmetrization.





combined with the definition of the Witt groups/spectrum implies that there is a splitting

$$KH_i((A,\varepsilon);\mathbb{Z}/q) \simeq K_i(A;\mathbb{Z}/q)^+ \oplus W_i((A,\varepsilon);\mathbb{Z}/q).$$

**Question 8.** Feng-Galatius-Venkatesh use the formalism of  $\Gamma$ -spaces to construct spectra from groupoids with a symmetric monoidal structure

- i) Read through Appendix A.3 to see how this works.
- ii) Explain how a Top-enriched groupoid with a symmetric monoidal structure gives rise to a  $\Gamma$ -space.
- iii) Explain why  $\Omega^{\infty} B^{\infty} M$  is the group completion of a monoid M in groupoids.

## References

[FGV20] Tony Feng, Soren Galatius, and Akshay Venkatesh. The galois action on symplectic k-theory, 2020.