Discussion section 5

Morgan Opie

Abstract

The following questions develop some ideas . Use them as jumping off points to explore ideas you find exciting or would like to understand better. They are in no particular order; feel free to jump around at your leisure.

Question 1. Recall that $\pi_i(X; \mathbb{Z}/q) := \pi_0 \mathrm{Map}_{\mathrm{Sp}}(\mathbb{S}^{i-1}/q, X)$, where $\mathbb{S}^n/q$ is the cofiber of the multiplication by q map on $\mathbb{S}^n$. Show by example that

i) Show by example that $\pi_i(-; \mathbb{Z}/q) \neq \pi_i(-) \otimes_{\mathbb{Z}} \mathbb{Z}/q$.

ii) Show that $\pi_i \mathrm{Map}_{\mathrm{Sp}}(-, \mathbb{S}^n/q \otimes H\mathbb{Z})$ computes $H\mathbb{Z}/q^i$.

Question 2. Read remark 3.4 of [FGV20] and understand the relationship between the Hodge/Betti maps as defined in section 3 and the classical Hodge bundle/Betti realization related to PPAVs.

Question 3. Deduce Corollary 3.6 from Theorem 3.5 [FGV20].

Question 4. Assuming $\mathrm{KSp}_i(\mathbb{Z}; \mathbb{Z}/q) \xrightarrow{\sim} K_i(\mathbb{Z}; \mathbb{Z}/q)^+ \oplus \pi_i(ku; \mathbb{Z}/q)^{(-)}$, deduce that $\mathrm{KSp}_{4k-2}(\mathbb{Z}; \mathbb{Z}/q) \simeq H^2\left(\mathrm{Spec}\left(\mathbb{Z}\left[\frac{1}{q}\right]\right)\right); \mu_q^{\otimes 2k}) \otimes \mathbb{Z}/q$.

Question 5. Example 3.9 of [FGV20] claims that the definition of the Hermitian K-theory as ε-quadratic forms implies that $KH(\mathbb{Z}, -1) = B^\infty(Q(\mathbb{Z}, -1))$ where $Q(\mathbb{Z}, -1)$ is the groupoid consisting of

- objects: a triple (L, ω, q) where L is a finitely-generated free $\mathbb{Z}$-module, $\omega : L \times L \to \mathbb{Z}$ is a skew-symmetric form, and q is a quadratic refinement of ω, i.e. $q : L/2 \to \mathbb{Z}/2$ such that $\omega(x, y) = q(x + y) - q(x) - q(y)$.
- a morphism $(L, \omega, q) \to (M, \eta, p)$ is an isomorphism of $\mathbb{Z}$-modules $f : L \xrightarrow{\sim} M$ such that $f^*(\eta) = \omega$.

Work this out in detail from the definition of $Q(A, \varepsilon)$.

Question 6. Show that the hyperbolization of a projective A-module has perfect symmetrization.

Question 7. Show that the factorization

$$\begin{array}{ccccccc} K_i(A;\mathbb{Z}/q) & & \xrightarrow{\quad\text{hyp}\quad} & KH_i((A,\varepsilon);\mathbb{Z}/q) & \xrightarrow{\quad\text{forget}\quad} & & K_i(A;\mathbb{Z}/q) \\ & \searrow & \overset{\exists}{\dashrightarrow} & & \overset{\exists}{\dashrightarrow} & \nearrow & \\ & & K_i(A;\mathbb{Z}/q)_{h\psi^{-1}} & & K_i(A;\mathbb{Z}/q)^{h\psi^{-1}} & & \end{array}$$

combined with the definition of the Witt groups/spectrum implies that there is a splitting

$$KH_i((A,\varepsilon);\mathbb{Z}/q) \simeq K_i(A;\mathbb{Z}/q)^+ \oplus W_i((A,\varepsilon);\mathbb{Z}/q).$$

Question 8. Feng-Galatius-Venkatesh use the formalism of Γ-spaces to construct spectra from groupoids with a symmetric monoidal structure

i) Read through Appendix A.3 to see how this works.

ii) Explain how a Top-enriched groupoid with a symmetric monoidal structure gives rise to a Γ-space.

iii) Explain why $\Omega^\infty B^\infty M$ is the group completion of a monoid M in groupoids.

References

[FGV20] Tony Feng, Soren Galatius, and Akshay Venkatesh. The galois action on symplectic k-theory, 2020.