Discussion section 3

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Abstract

The following questions expand on some concepts introduced in the last two weeks. Use them as jumping off points to explore ideas you find exciting or would like to understand better. They are in no particular order; feel free to jump around at your leisure.

Question 1. (From 18.785.)

1. Show that an absolute value on a field k is nonarchimedian iff the $|n| \leq 1$ for all n, where we use n to denote both an integer and its image under the unique morphism $Z \to k$.
2. Prove Ostrowski's theorem.

Question 2. (From Math 223)

1. Let K be a number field and H_K/K be its Hilbert class field. Prove that $H_K/\mathbb{Q}$ is a Galois extension.
2. Now suppose $K/\mathbb{Q}$ is Galois. We thus we have a short exact sequence

$$0 \to \mathrm{Gal}(H_K/K) \to \mathrm{Gal}(H_K/\mathbb{Q}) \to \mathrm{Gal}(K/\mathbb{Q}) \to 0. \quad (1)$$

   Show that we get a well-defined $\mathrm{Gal}(K/\mathbb{Q})$-action on $\mathrm{Gal}(H_K/K)$ by lifting and conjugating.
3. By considering the Galois action on Frobenius, show that the Artin isomorphism

$$\mathrm{Cl}_K \xrightarrow{\sim} \mathrm{Gal}(H_K/K)$$

   is equivariant with respect to the $\mathrm{Gal}(K/\mathbb{Q})$-action.
4. Show that if $K/\mathbb{Q}$ is imaginary quadratic then we can complex conjugation to (non-canonically) split (1). In general it will not be split, but counterexamples may be annoying to concoct (see MO).
5. (for people who more about class field theory) extensions of the form (1) correspond to classes in $H^2(\mathrm{Gal}(L/K), \mathrm{Cl}_K)$, and the above comes from the fundamental class $[u] \in H^2(\mathrm{Gal}(L/K), C_L)$. Use this to show when $K/\mathbb{Q}$ is cyclic, (1) splits.

6. (for people who know more about class field theory) Extensions of the form (1) correspond to classes in $H^2(\mathrm{Gal}(K/Q), \mathrm{Cl}_K)$. Use this to show when $K/\mathbb{Q}$ is cyclic, (1) splits (Hint: what can you say specifically about the class (1) in terms of class field theory?).