Discussion section 2

Natalia Pacheco-Tallaj and Lucy Yang

February 22nd, 2021

Abstract

The following questions expand on some concepts introduced in the last two weeks. Use them as jumping off points to explore ideas you find exciting or would like to understand better. They are in no particular order; feel free to jump around at your leisure.

Question 1. A functor $F : \mathcal{C} \to \mathcal{D}$ of ∞-categories (not necessarily stable) is called *excisive* if it takes pushout squares to pullback squares. If $\mathcal{C}$ and $\mathcal{D}$ have zero objects, then F is *reduced* if it preserves zero objects.

[Lur17] defines spectrum objects of $\mathcal{C}$ to be reduced excisive functors from finite pointed spaces $\mathrm{Exc}_*(\mathcal{S}\mathrm{pc}_*^{\mathrm{fin}}, \mathcal{C})$. Taking $\mathcal{C} = \mathcal{S}\mathrm{pc}_*$ gives us the category of spectra. Write down functors between this definition of spectrum objects to the category Dylan introduced in his talk, i.e. a spectrum as a series of spaces $(E_n)_{n\in\mathbb{N}}$ with equivalences $E_n \to \Omega E_{n+1}$.

Question 2. Look at/compare the various definitions/constructions of the smash product on spectra [Lur17, 4.8.2 (S3)] or referenced here.

Question 3. A model for the *mapping cone/homotopy colimit* of a map of spaces $f : X \to Y$ is $Cf := X \times [0,1] \sqcup Y/((x,0) \sim (x',0)$ and $(x,1) \sim f(x))$. There is a natural inclusion $i : Y \to Cf$.

The *(1)-categorical colimit* is defined to be $Y/f(x) \sim f(x')$.

1. Show that the data of a map $Cf \to Z$ is the same as the data of a map $Y \xrightarrow{g} Z$ and a nullhomotopy of $g \circ i$.
2. Are the homotopy and (1)-categorical colimits always (homotopy) equivalent? Prove or give a counterexample.

Question 4. Let M be an $\mathbb{E}_\infty$-monoid. Show that $\pi_0(M^{\mathrm{gp}}) = (\pi_0 M)^{\mathrm{gp}}$, so that K_0 as defined in Lucy's talk agrees with the classical definition of the Grothendieck group.

Question 5. Show that any ε-quadratic R-module is an orthogonal direct summand of $H(R)^{\oplus n}$ for some $n \in \mathbb{Z}_{>0}$.

Question 6. Let (X, x) be a connected pointed space, and $E \leq \pi_1(X, x)$ a perfect subgroup. Show that there exists a universal map of pointed spaces $(X, x) \to (Y, y)$ which induces an isomorphism on homology and exhibits $\pi_1(Y, y)$ as the quotient $\pi_1(X, x)/E$ [Hat01, p.374]. *Hint:* Try showing this in the case where $H_1(X) = 0$ and $E = \pi_1(X, x)$ first.

References

[Hat01] Allen Hatcher. *Algebraic Topology*. 2001.

[Lur17] Jacob Lurie. *Higher algebra*. 2017.