Discussion section 1

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Abstract

The following questions hint at what will be covered in the coming weeks. Use them as jumping off points to explore ideas you find exciting or would like to understand better. They are in no particular order; feel free to jump around at your leisure.

1 Towards stable homotopy theory

Question 1. Let X, Y be pointed spaces which are m, n-connected[1], respectively. Show that the natural map $X \wedge Y \to X \times Y$ is $(m+n)$-connected.

Question 2. Define the $\mathbb{E}_n$-operad to have kth space $\mathbb{E}_n(k)$ the space of rectilinear embeddings of k disjoint n-cubes $([0,1]^n)^{\sqcup k}$ into the n-cube $[0,1]^n$. Operadic composition is defined to be composition of embeddings.

An example of an $\mathbb{E}_n$-algebra is (your favorite model of) the nth (based) loop space of a pointed space $\Omega^n X := \mathrm{Map}_*((S^n,*),(X,x)) \simeq \{f : [0,1]^n \to X \mid f|_{\partial[0,1]^n} \equiv *\}$.

1. What is the difference between being an $\mathbb{E}_1$-algebra and a strictly associative algebra (i.e. an algebra over the associative operad, which has $Assoc(2) = \{*\}$)?
2. We can define the *Moore loop space* of a pointed space (X, x) by $\Omega' X := \{f : [0,r] \to X \mid f(0) = f(r)\}$.
   (a) Show that $\Omega' X$ admits a *strictly associative* multiplication.
   (b) Show/convince yourself that ΩX and $\Omega' X$ are homotopy equivalent *as* $\mathbb{E}_1$*-algebras.*
3. Recall that the higher homotopy groups $\pi_n X$ are commutative for $n \geq 2$. Is [pick your favorite model of $\Omega^2 X$] equivalent to space with a strictly commutative multiplication?

Question 3. Show a symmetric monoidal category can be interpreted as an n-category such that any two parallel j-morphisms are equivalent for $j < n-1$, for any $n \geq 4$.

[1] A space X is k-connected if $\pi_n X = 0$ for all $n \leq k$

Question 4. Show the geometric realization of the one object category with automorphism group G gives you a space homotopy equivalent to BG as obtained from the bar construction.

2 Follow-ups to the first talk

Question 5. Show the space of almost-complex structures on a symplectic vector space/manifold is contractible, and hence we have a weak homotopy equivalence between ku and $K\mathrm{Sp}(\mathbb{R})$, whatever those are [FGV20, 3.2.2].

Question 6. Show if $\mathfrak{p}$ is principal, the polarization on $\mathbb{C}^g/\mathfrak{p}$ gives a trivial symplectic K-theory class. If this is tricky, start with a specific case, e.g. $q = 3, g = 2$.

Question 7. Show K_0 of a field is $\mathbb{Z}$.

References

[FGV20] Tony Feng, Soren Galatius, and Akshay Venkatesh. The galois action on symplectic k-theory, 2020.