CM Classes Exhaust Symplectic K-Theory

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Goal

The goal of today's talk is to prove the following theorem

Theorem (5.1 in FGV)

The map

$$\mathcal{P}^{-}_{K_q} \xrightarrow{\mathrm{ST}} \mathcal{A}_g(\mathbb{C}) \xrightarrow{\varphi} \mathrm{SP}(\mathbb{Z})$$

induces a surjective map

$$\gamma\colon \pi^s_{4k-2}(|\mathcal{P}^{-}_{K_q}|; \mathbb{Z}/q) \to K\mathrm{Sp}_{4k-2}(\mathbb{Z}; \mathbb{Z}/q)$$

for all $k \geq 1$.

Conventions

- $q = p^n$ is a power of an odd prime
- $\mathcal{O}_q = \mathbb{Z}[e^{2\pi i/q}]$
- $K_q = \mathcal{O}_q \otimes \mathbb{Q}$

Outline of Proof

- Step 1. Get the map γ from the composite $\varphi \circ \mathrm{ST}$.
- Step 2a Recast the domain of γ as

$$\pi_*^S(|\mathcal{P}_{K_q}^-|; \mathbb{Z}/q) \simeq \pi_*^S(BU_1(\mathcal{O}_q); \mathbb{Z}/q) \otimes \mathbb{Z}[\pi_0(\mathcal{P}_{K_q}^-)]$$

- Step 2b Recast the codomain using the Hodge map c_H and the Betti map c_B from Morgan's talk,

$$K\mathrm{Sp}_{4k-2}(\mathbb{Z}; \mathbb{Z}/q) \xrightarrow{\sim} \pi_{4k-2}(ku; \mathbb{Z}/q)^{(-)} \times K_{4k-2}(\mathbb{Z}; \mathbb{Z}/q)^{(+)}$$

- These reduce the main theorem to showing that γ composed with c_H and with c_B are surjective.
- Step 3. Show that $c_H \circ \gamma$ is surjective.
- Step 4. Show that $c_B \circ \gamma$ is surjective.

Step 1

Consider the composite $\varphi \circ \mathrm{ST}$

$$\mathcal{P}^{-}_{K_q} \to \mathrm{SP}(\mathbb{Z})$$

Taking associated spaces,

$$|\mathcal{P}^{-}_{K_q}| \to |\mathrm{SP}(\mathbb{Z})|$$

Post-compose with the group completion map

$$|\mathcal{P}^{-}_{K_q}| \to |\mathrm{SP}(\mathbb{Z})| \to \Omega^{\infty} K\mathrm{Sp}(\mathbb{Z})$$

Taking the adjoint,

$$\Sigma^{\infty}_{+}|\mathcal{P}^{-}_{K_q}| \to K\mathrm{Sp}(\mathbb{Z})$$

Finally, taking $\mathbb{Z}/q$ homotopy groups gives

$$\gamma \colon \pi^{s}_{4k-2}(|\mathcal{P}^{-}_{K_q}|; \mathbb{Z}/q) \to K\mathrm{Sp}_{4k-2}(\mathbb{Z}; \mathbb{Z}/q)$$

Step 2a

Recall that an object of $\mathcal{P}^{-}_{K_q}$ is a pair (L, b) where L is a projective $\mathcal{O}_q$-module of rank 1 and b is skew-Hermitian form on L valued in $\mathrm{Hom}_{\mathbb{Z}}(\mathcal{O}_q, \mathbb{Z})$.
Morphisms in the groupoid $\mathcal{P}^{-}_{K_q}$ from $(L, b) \to (L', b')$ are $\mathcal{O}_q$-linear isomorphisms $\phi\colon L \to L'$ such that

$$b(\phi(x), \phi(y)) = b(x, y)$$

for all $x, y \in L$.

Lemma

The automorphism group of (L, b) *in* $\mathcal{P}^{-}_{K_q}$ *is*

$$\mathrm{Aut}(L, b) = \{u \in \mathcal{O}_q : u\bar{u} = 1\} = U_1(\mathcal{O}_q)$$

Proof.

Since L is rank 1, a $\mathcal{O}_q$-linear isomorphism $\phi\colon L \to L$ must by multiplication by some $u \in \mathcal{O}_q$. For any $x, y \in L$, we have

$$b(\phi(x), \phi(y)) = b(u \cdot x, u \cdot y) = u\bar{u}b(x, y)$$

Since ϕ satisfies $b(\phi(x), \phi(y)) = b(x, y)$, we must have $u\bar{u} = 1$. □

The associated space for the groupoid $\mathcal{P}^-_{K_q}$ is then homotopy equivalent to the classifying space on the automorphism group of any object, cross with the connected components,

$$|\mathcal{P}^-_{K_q}| \simeq BU_1(\mathcal{O}_q) \times \pi_0(\mathcal{P}^-_{K_q})$$

Taking $\mathbb{Z}/q$-homotopy groups,

$$\pi^s_*(|\mathcal{P}^-_{K_q}|; \mathbb{Z}/q) \simeq \pi^s_*(BU_1(\mathcal{O}_q); \mathbb{Z}/q) \otimes \mathbb{Z}[\pi_0(\mathcal{P}^-_{K_q})]$$

Consider the map $\mathbb{Z}/q \to \mathcal{O}_q^\times$ sending $a \mapsto e^{2\pi i a/q}$.
Get an induced map

$$\pi_2^s(B\mathbb{Z}/q; \mathbb{Z}/q) \to \pi_2^s(BU_1(\mathcal{O}_q); \mathbb{Z}/q)$$

There is a polynomial subalgebra in the LHS generated by the Bott element β.

Specific Classes that Surject

We will show that classes of the form

$$\beta^{2k-1} \otimes [(L, b)] \in \pi^s_{4k-2}(|\mathcal{P}^-_{K_q}|; \mathbb{Z}/q)$$

for $(L, b) \in \mathcal{P}^-_{K_q}$ map, under γ, to generators for $K\mathrm{Sp}_{4k-2}(\mathbb{Z}; \mathbb{Z}/2)$.

$$\begin{tikzcd}
\pi^s_*(B\mathbb{Z}/q;\mathbb{Z}/q)\times\pi_0(\mathcal{P}^-_{K_q}) \arrow[r] & \pi^s_*(BU_1(\mathcal{O}_q);\mathbb{Z}/q)\times\pi_0(\mathcal{P}^-_{K_q}) \arrow[d, "\cong"] \\
 & \pi^s_{4k-2}(|\mathcal{P}^-_{K_q}|;\mathbb{Z}/q) \arrow[d, "\gamma"] \\
 & K\mathrm{Sp}_{4k-2}(\mathbb{Z};\mathbb{Z}/q)
\end{tikzcd}$$

The classes $\beta^{2k-1}\otimes[(L,b)]$ live in the upper left corner. We want to study their image under the composite.

Step 2b

We'll use the following theorem from Morgan's talk to better understand the codomain of γ.

Theorem (3.5 in FGV)

The Hodge and Betti maps (c_H, c_B) *define an isomorphism*

$$(c_H, c_B)\colon K\mathrm{Sp}_{4k-2}(\mathbb{Z}; \mathbb{Z}/q) \to \pi_{4k-2}(ku; \mathbb{Z}/q)^{(-)} \times K_{4k-2}(\mathbb{Z}; \mathbb{Z}/q)^{(+)}$$

The $\pm$ here denote the (± 1) eigenspaces for the Adams operation ψ^{-1} on $K(\mathbb{Z}/\mathbb{Z}/q)$.

Restatement of Theorem

We have reduced the goal of this talk to proving the following:
As $[(L,b)] \in \pi_0(\mathcal{P}^{-}_{K_q})$ varies,

- $\{c_H \circ \gamma(\beta^{2k-1} \otimes [(L,b)])\}$ generate $\pi_{4k-2}(ku; \mathbb{Z}/q)^{(-)}$
- $\{c_B \circ \gamma(\beta^{2k-1} \otimes [(L,b)])\}$ generate $K_{4k-2}(\mathbb{Z}; \mathbb{Z}/q)^{(+)}$.

Step 3: $c_H \circ \gamma$ is surjective

Recall that the Hodge map c_H comes from a zigzag of maps/equivalences

$$SP(\mathbb{Z}) \to SP(\mathbb{R}^{\mathrm{top}}) \xleftarrow{\simeq} \mathcal{U}(\mathbb{C}^{\mathrm{top}})$$

We're looking at the diagram

$$\begin{tikzcd}
\mathcal{P}^{-}_{K_q} \arrow[r, "\varphi \circ ST"] & SP(\mathbb{Z}) \arrow[d] \\
 & SP(\mathbb{R}^{\mathrm{top}}) \\
 & \mathcal{U}(\mathbb{C}^{\mathrm{top}}) \arrow[u]
\end{tikzcd}$$

(From last week) the composite $\varphi \circ ST \colon \mathcal{P}^-_{K_q} \to SP(\mathbb{Z})$ sends a pair (L, b) to the pair $(L_\mathbb{Z}, -\mathrm{Tr}^{K_q}_{\mathbb{Q}}(b(-,-)))$.
Basically because $\mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$, this assignment descends to a map out of $\mathcal{P}^-_{K_q \otimes \mathbb{R}}$.

$$\begin{tikzcd}
\mathcal{P}^-_{K_q} \arrow[r, "\varphi\circ ST"] \arrow[d] & SP(\mathbb{Z}) \arrow[d] \\
\mathcal{P}^-_{K_q\otimes\mathbb{R}} \arrow[r] & SP(\mathbb{R}^{\mathrm{top}}) \\
 & \mathcal{U}(\mathbb{C}^{\mathrm{top}}) \arrow[u]
\end{tikzcd}$$

We want a lift

$$\begin{tikzcd}
\mathcal{P}^-_{K_q} \arrow[r, "\varphi\circ ST"] \arrow[d] & SP(\mathbb{Z}) \arrow[d] \\
\mathcal{P}^-_{K_q\otimes\mathbb{R}} \arrow[r] \arrow[dr, dashed] & SP(\mathbb{R}^{\mathrm{top}}) \\
 & \mathcal{U}(\mathbb{C}^{\mathrm{top}}) \arrow[u]
\end{tikzcd}$$

so we can compute $c_H \circ \gamma$ by going the other way around the diagram.

We'll take a lift coming from inverting the weak equivalence $\mathcal{U}(\mathbb{C}^{\mathrm{top}}) \to SP(\mathbb{R}^{\mathrm{top}})$. This is equivalent to constructing, given a symplectic real vector space, a complex structure and Hermitian metric compatible with the given symplectic form as the imaginary part.

- The CM structure (L, b) on K_q incudes this structure on $L_{\mathbb{R}}$.
- $\Phi_{(L,b)} = \{j\colon K_q \to \mathbb{C}\colon \mathrm{Im}(jb(x,x)) \geq 0$ for all $x \in L_{\mathbb{R}}\}$

So, we want to compute the image of $\beta^{2k-1} \otimes [(L, b)]$ under the composite

$$\begin{tikzcd}
\pi^s_*(|\mathcal{P}^-_{K_q}|;\mathbb{Z}/q) \arrow[r] \arrow[d] \arrow[rr, bend left, "\gamma"] & \pi^s_*(|SP(\mathbb{Z});\mathbb{Z}/q) \arrow[r] \arrow[d] & K\mathrm{Sp}(\mathbb{Z};\mathbb{Z}/q) \arrow[dd, "c_H"] \\
\pi^s_*(|\mathcal{P}^-_{K_q\otimes\mathbb{R}}|;\mathbb{Z}/q) \arrow[r] \arrow[dr] & \pi^s_*(|SP(\mathbb{R}^{\mathrm{top}});\mathbb{Z}/q) & \\
 & \pi^s_*(|\mathcal{U}(\mathbb{C}^{\mathrm{top}})|;\mathbb{Z}/q) \arrow[u] \arrow[r] & \pi_*(ku;\mathbb{Z}/q)
\end{tikzcd}$$

Let $(L, b) \in \mathcal{P}^{-}_{K_q}$. Consider the full groupoid $\{(L, b)\}//\mathrm{Aut}(L, b)$ of (L, b) in $\mathcal{P}^{-}_{K_q}$. We want to describe the composite

$$\{(L, b)\}//\mathrm{Aut}(L, b) \hookrightarrow \mathcal{P}^{-}_{K_q} \to \mathcal{U}(\mathbb{C}^{\mathrm{top}})$$

- Such a map is equivalent to the data of a unitary representation of $\mathrm{Aut}(L, b)$.
- For each $j\colon K_q \to \mathbb{C}$ in $\Phi_{(L,b)}$, we get a 1d unitary representation by restricting to $\mathrm{Aut}(L, b)$,

$$\begin{tikzcd}
K_q \arrow[r, "j"] & \mathbb{C} \\
\mathrm{Aut}(L, b) \arrow[u] \arrow[r] & U_1(\mathbb{C}) \arrow[u]
\end{tikzcd}$$

Lemma

The composition

$$\{(L,b)\}//\mathrm{Aut}(L,b) \hookrightarrow \mathcal{P}^{-}_{K_q} \to \mathcal{P}^{-}_{K_q\otimes\mathbb{R}} \to \mathcal{U}(\mathbb{C}^{\mathrm{top}})$$

corresponds to the unitary representation

$$\bigoplus_{j\in\Phi_{(L,b)}} j|_{\mathrm{Aut}(L,b)}$$

- Makes sense because the lift $\mathcal{P}^{-}_{K_q\otimes\mathbb{R}} \to \mathcal{U}(\mathbb{C}^{\mathrm{top}})$ was defined using $\Phi_{(L,b)}$.

Image of Bott Element

An element $j \in \Phi_{(L,b)}$ is determined by an embedding $j\colon \mathcal{O}_q \to \mathbb{C}$. Embeddings

$$\mathcal{O}_q = \mathbb{Z}[e^{2\pi i/q}] \to \mathbb{C}$$

are parameterized by $s \in (\mathbb{Z}/q)^\times$,

$$j_s(e^{2\pi i/q}) = e^{2\pi is/q}$$

- By Dylan's talk, we have an induced map

  $$(j_s)_*\colon \pi_*^s(B\mathbb{Z}/q;\mathbb{Z}/q) \to \pi_*(ku;\mathbb{Z}/q)$$

  sending the Bott element β to the mod q Bott element Bott.
- So $(j_s)_*(\beta^i) = s^i \mathrm{Bott}^i$.

Image of $\beta^i[(L,b)]$ under $c_H \circ \gamma$

Proposition (5.2 in FGV)

Let $(L,b) \in \mathcal{P}^-_{K_q}$. The image of $\beta^i[(L,b)]$ under $c_H \circ \gamma$ is

$$\sum_{\substack{s \in (\mathbb{Z}/q)^\times \\ s.t.\ j_s \in \Phi_{(L,b)}}} s^i \mathrm{Bott}^i \in \pi_{2i}(ku; \mathbb{Z}/q)$$

- *Moreover, for odd i, there exists a CM structure Φ on K_q for which the above element is a generator of $\pi_{2i}(ku; \mathbb{Z}/q) \cong \mathbb{Z}/q$.*
- *Thus $c_H \circ \gamma$ is surjective.*

Proof.

- We just need to prove the second claim.
- The element Bott^i generates $\pi_{2i}(ku; \mathbb{Z}/q)$, so just need coefficients to be a unit.
- A CM structure Φ on K_q is the same as a choice of subset $X \subset (\mathbb{Z}/q)^\times$ containing exactly one element of each set $\{a, -a\}$.
- Choose any such X and let X' be the same as X but with ± 1 replaced with ∓ 1.
- Then

$$\sum_{\substack{s\in(\mathbb{Z}/q)^\times \\ \text{s.t. } s\in X}} s^i - \sum_{\substack{s\in(\mathbb{Z}/q)^\times \\ \text{s.t. } s\in X'}} s^i = (-1)^i - (1)^i = 2$$

so at least one is a unit.

$\square$

Step 4: $c_B \circ \gamma$ is surjective

Recall that the Betti map

$$c_H\colon K\mathrm{Sp}_{4k-2}(\mathbb{Z};\mathbb{Z}/q) \to K_{4k-2}(\mathbb{Z};\mathbb{Z}/q)^{(+)}$$

comes from group completing and taking homotopy groups on the forgetful map

$$SP(\mathbb{Z}) \to P(\mathbb{Z})$$

The map γ comes from group completing and taking homotopy groups of the composite

$$\mathcal{P}^{-}_{K_q} \xrightarrow{ST} \mathcal{A}_g(\mathbb{C}) \to SP(\mathbb{Z})$$

that sends a pair (L, b) to

$$(L, b) \mapsto (L_{\mathbb{R}}/L_{\mathbb{Z}}, \mathcal{L}) \mapsto (H_1(L_{\mathbb{R}}/L_{\mathbb{Z}}; \mathbb{Z}), c_1(\mathcal{L})) = (L_{\mathbb{Z}}, c_1(\mathcal{L}))$$

- So the composite of the Betti map and γ sends (L, b) to $L_{\mathbb{Z}}$.

Another way to obtain $L_{\mathbb{Z}}$ from (L, b) is follow the forgetful map

$$\mathcal{P}^{-}_{K_q} \to \mathrm{Pic}(\mathcal{O}_q)$$

sending (L, b) to L, and then do the composite

$$\mathrm{Pic}(\mathcal{O}_q) \to P(\mathcal{O}_q) \to P(\mathbb{Z})$$

- So we get a commutative diagram

$$\begin{tikzcd}
\mathcal{P}^{-}_{K_q} \arrow[r] \arrow[d] & SP(\mathbb{Z}) \arrow[d] \\
\mathrm{Pic}(\mathcal{O}_q) \arrow[r] & P(\mathbb{Z})
\end{tikzcd}$$

Taking associated spaces, group completing, adjointing, and taking homotopy groups we get a diagram

$$\begin{tikzcd}
\pi_*^s(|\mathcal{P}_{K_q}^-|; \mathbb{Z}/q) \arrow[r, "\gamma"] \arrow[d] & K\mathrm{Sp}(\mathbb{Z}; \mathbb{Z}/q) \arrow[d, "c_B"] \\
\pi_*^s(|\mathrm{Pic}(\mathcal{O}_q)|; \mathbb{Z}/q) \arrow[r, "\mathrm{tr}"'] & K(\mathbb{Z}; \mathbb{Z}/q)
\end{tikzcd}$$

- The bottom horizontal arrow is the transfer.
- The Betti maps actually lands in the $(+1)$-eigenspace of the Adams operation ψ^{-1} on $K(\mathbb{Z}; \mathbb{Z}/q)$. So we can replace the bottom right corner with $K(\mathbb{Z}; \mathbb{Z}/q)^{(+)}$.

Corollary (5.4 in FGV)

The composite

$$\pi^s_{4k-2}(|\mathcal{P}^-_{K_q}|;\mathbb{Z}/q) \xrightarrow{\gamma} K\mathrm{Sp}_{4k-2}(\mathbb{Z};\mathbb{Z}/q) \xrightarrow{c_B} K_{4k-2}(\mathbb{Z};\mathbb{Z}/q)^{(+)}$$

sends $\beta^{2k-1}[(L,b)]$ *to*

$$\mathrm{tr}(\beta^{2k-1}([L]-1))$$

Proof.

- Recall that we defined the element $\beta^{2k-1}[(L,b)]$ using the equivalence

$$\pi_*^s(|\mathcal{P}_{K_q}^-|;\mathbb{Z}/q) \simeq \pi_*^s(BU_1(\mathcal{O}_q);\mathbb{Z}/q) \otimes \mathbb{Z}[\pi_0(\mathcal{P}_{K_q}^-)]$$

- We have a similar equivalence

$$\pi_*^s(|\mathrm{Pic}(\mathcal{O}_q)|;\mathbb{Z}/q) \simeq \pi_*^s(B\mathcal{O}_q^\times;\mathbb{Z}/q) \otimes \mathbb{Z}[\pi_0(\mathrm{Pic}(\mathcal{O}_q))]$$

- The map $\mathcal{P}_{K_q}^- \to \mathrm{Pic}(\mathcal{O}_q)$ works nicely with these equivalences so that we can pull out the Bott class part, $\beta^{2k-1}[(L,b)] = \beta^{2k-1}[L]$
- Thus $c_B(\gamma(\beta^{2k-1}[(L,b)])) = \mathrm{tr}(\beta^{2k-1}[L])$
- It turns out that $\mathrm{tr}(\beta^{2k-1})$ lands in the (-1)-eigenspace of ψ^{-1}, and so vanishes. Thus we can rewrite this as

$$\mathrm{tr}(\beta^{2k-1}([L]-1))$$

Conclusion

Proposition

The map $c_B \circ \gamma$ is surjective.

Proof.

By Proposition 2.17 in FGV, classes of the form $\mathrm{tr}(\beta^{2k-1}([L]-1))$ generate $K_{4k-2}(\mathbb{Z};\mathbb{Z}/q)^{(+)}$ as $[L]$ ranges over $\pi_0(\mathrm{Pic}(\mathcal{O}_q))$. For every such L, can cleverly use the Lemma stated at the end of last talk (Prop. 4.7 in FGV) to show that there exists some $(L', b) \in \mathcal{P}^{-}_{K_q}$ so that

$$\mathrm{tr}(\beta^{2k-1}([L']-1))) = \mathrm{tr}(\beta^{2k-1}([L]-1)))$$

□