CM Abelian Varieties

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Goals

- Define CM abelian varieties and polarizations
- Show how abelian varieties induce K-theory classes
- Show how to construct/parametrize CM abelian varieties
- (Next week) show that CM abelian varieties generate K-theory

Abelian varieties

Definition
An *abelian variety* over $\mathbb{C}$ is a group variety which is a connected complex projective variety.

Proposition
An abelian variety is in fact an abelian group object in the category of varieties.

Example: elliptic curves

- Any smooth projective complex genus 1 curve with a given rational point O is naturally an abelian variety (an *elliptic curve*)
- Elliptic curves over $\mathbb{C}$ are isomorphic as abelian varieties to $\mathbb{C}/L$ for a lattice L
- In fact, any complex abelian variety is of the form $\mathbb{C}^g/L$ (though not all full-rank L work!)

Riemann forms

Let $V \cong \mathbb{C}^g$ and $\mathbb{Z}^{2g} \cong L \subset V$ a lattice of rank $2g$.

Definition

A skew-symmetric form $E : L \times L \to \mathbb{Z}$ is a *Riemann form* if the extension $E_{\mathbb{R}} : V \times V \to \mathbb{R}$ satisfies for all $v, w \in V$

(a) $E_{\mathbb{R}}(iv, iw) = E_{\mathbb{R}}(v, w)$, and

(b) If $v \neq 0$, then $E_{\mathbb{R}}(iv, v) > 0$.

When condition (a) holds,

$$H(v, w) = E_{\mathbb{R}}(iv, w) + iE(v, w)$$

defines a Hermitian form on V. Then condition (b) says H is positive definite.

Riemann forms

Theorem

A complex torus V/L is (the analytification of) an abelian variety over $\mathbb{C}$ iff L admits a Riemann form. Equivalently, there is a positive-definite Hermitian form H on V such that $\Im H(L, L) \subset \mathbb{Q}$.

In this case $H_1(A(\mathbb{C})^{\mathrm{an}}; \mathbb{Z}) \cong L$ has a skew-symmetric form. This is a class in

$$H^2(A(\mathbb{C})^{\mathrm{an}}; \mathbb{Z}) \cong \Lambda^2 L^*.$$

When the form is a Riemann form, this class is $c_1(\mathcal{L})$ for an ample line bundle $\mathcal{L}$ on A.

The dual abelian variety

Let $t_a : A \to A$ denote translation by $a \in A(\mathbb{C})$. The *dual abelian variety* $A^\vee$ of A is the variety classifying line bundles on A, trivialized at 0, such that $t_a^*\mathcal{L} \cong \mathcal{L}$. (This is the *Picard scheme* of A.)

By the classifying property, $A \times A^\vee$ has a universal line bundle $\mathcal{P}$, the *Poincare bundle*.

Dualization gives an equivalence $\mathbf{AbVar}^{\mathrm{op}} \to \mathbf{AbVar}$ and satisfies $A^{\vee\vee} \cong A$.

In the complex case, $A \cong V/L$. We can take $A^\vee$ to be V^*/L^*.

Polarizations

Proposition

Let $\mathcal{L}$ be an ample line bundle on an abelian variety A. Then

$$\lambda_{\mathcal{L}} : A \to A^{\vee}$$

$$x \mapsto t_x^* \mathcal{L} \otimes \mathcal{L}^{-1}$$

is an isogeny.

Recall that an *isogeny* is a surjection of algebraic groups with finite kernel.
An isogeny of the form $\lambda_{\mathcal{L}}$ for some ample $\mathcal{L}$ is called a *polarization* of A.
An isogeny with kernel 0 is an isomorphism, and in this case we call $\lambda_{\mathcal{L}}$ a *principle polarization*.

Polarizations

There are surjective maps:

$$\begin{array}{c}\text{Line bundles on } A \\ \Downarrow \\ \text{Self-dual morphisms } A \to A^\vee \\ \Downarrow \\ \text{Skew-symmetric bilinear forms on } H_1(A(\mathbb{C})^{\mathrm{an}}; \mathbb{Z})\end{array}$$

The composite is the Chern class $c_1(\mathcal{L}) \in H^2(A(\mathbb{C})^{\mathrm{an}}; \mathbb{Z})$.

Polarizations

This upgrades to:

$$\begin{gathered}\text{Ample line bundles on } A \\ \Downarrow \\ \text{Polarizations } A \to A^\vee \text{ (up to isogeny)} \\ \Updownarrow \\ \text{Riemann forms on } H_1(A(\mathbb{C})^{\mathrm{an}}; \mathbb{Z}) \\ \text{(up to overall rational factor)}\end{gathered}$$

Define a correspondence between abelian varieties A and B to be a line bundle on $A \times B$ with trivializations on $A \times \{0\}$ and $\{0\} \times B$ that agree at the origin.
Then polarizations are furthermore in bijection with symmetric correspondences on $A \times A$ which pull back on the diagonal to an ample bundle on A.

This is a "symmetric positive-definite bilinear form" on A.

The moduli stack of abelian varieties

The *moduli stack of principally polarized abelian varieties* $\mathcal{A}_g$ classifies complex abelian varieties A of dimension g, equipped with a principle polarization $\lambda : A \to A^\vee$.

The moduli stack of abelian varieties

A complex PPAV is specified by a full-rank lattice $L \subset \mathbb{C}^g$ with an identification $\lambda : \mathbb{C}^g/L \xrightarrow{\sim} (\mathbb{C}^g)^*/L^*$. By principleness this gives $L \xrightarrow{\sim} L^*$. Choosing L to contain the real basis $e_1, ..., e_g$ of $\mathbb{C}^g$, then L and λ are specified by the lift of λ,

$$\tau : \mathbb{C}^g \to (\mathbb{C}^g)^*$$

which, by the properties of the Riemann form, satisfies

$$\tau^T = \tau \quad \text{and} \quad \Im\tau \text{ is positive definite.}$$

Let $\mathbb{H}_g$ denote the space of such τ (the *Siegel upper half-space*). Change of basis preserving the Riemann form preserves the underlying variety and polarization, so

$$\mathcal{A}_g(\mathbb{C}) \cong \mathbb{H}_g/\mathrm{Sp}_{2g}(\mathbb{Z})$$

The map to K-theory

Let $\mathcal{L}_\lambda$ be the pullback of the Poincare bundle under the graph morphism $A \to A \times A^\vee$. The map

$$\begin{array}{ccc} \mathcal{A}_g(\mathbb{C}) & \longrightarrow & \mathcal{SP}(\mathbb{Z}) \\ (A, \lambda) & \longmapsto & (H_1(A(\mathbb{C})^{\mathrm{an}}; \mathbb{Z}), c_1(\mathcal{L}_\lambda)) \end{array}$$

extends to a map of groupoids.

So we get

$$|\mathcal{A}_g(\mathbb{C})| \to |\mathcal{SP}(\mathbb{Z})| \to \Omega^\infty \mathrm{KSp}(\mathbb{Z})$$

and adjointly

$$\Sigma^\infty_+ |\mathcal{A}_g(\mathbb{C})| \to \mathrm{KSp}(\mathbb{Z}).$$

We wish to show this map is surjective on (mod q) homotopy. In fact, there is a nice subgroupoid of $\mathcal{A}_g(\mathbb{C})$ for which this is true.

Complex multiplication

We wish to construct maps $\Sigma^\infty_+(B(\mathbb{Z}/q)) \to \mathrm{KSp}(\mathbb{Z})$ (to get "CM classes" in $\mathrm{KSp}_*(\mathbb{Z}; \mathbb{Z}/q)$ as images of the Bott element).

We get these from maps $B(\mathbb{Z}/q) \to \mathcal{A}_g(\mathbb{C})$, equiv. $\mathbb{Z}/q$ actions on PPAVs.

Such an action can be realized in PPAVs admitting an action of $\mathcal{O}_q \doteq \mathbb{Z}[\zeta_q]$. These are principally polarized abelian varieties with *complex multiplication*.

Complex multiplication

Definition

Let A be an abelian variety of dimension g. If $\text{End}(A) \otimes \mathbb{Q}$ has a commutative $\mathbb{Q}$-subalgebra of dimension $2g$, then A is said to have *complex multiplication*.

What do these look like?

CM orders and CM fields

Definition

A *CM field* E is a number field which is a totally imaginary extension of a totally real field E_+.
($E = E_+(\sqrt{d})$, where every algebra map $E_+ \to \mathbb{C}$ has real image and takes d to a negative number.)

A *CM algebra* is a product of CM fields.
A *CM order* is an order in a CM algebra which is stable under complex conjugation.

An *order* in E is a full rank integral sublattice; a free $\mathbb{Z}$-subalgebra $\mathcal{O}$ such that $\mathcal{O} \otimes \mathbb{Q} \cong E$.

If A has complex multiplication, then $\mathrm{End}(A)$ is a CM order.

Constructing CM abelian varieties

A PPAV with complex multiplication is specified by the following data:

1. $\mathcal{O}$ a CM order (with CM algebra E)
2. $\mathfrak{a}$ an $\mathcal{O}$-submodule of E such that $\mathfrak{a} \otimes \mathbb{Q} \cong E$
3. $\Phi : \mathcal{O} \otimes \mathbb{R} \xrightarrow{\sim} \mathbb{C}^g$ an algebra isomorphism
4. A purely imaginary element u in E

(1-3) give $\mathcal{O} \otimes \mathbb{R}/\mathfrak{a}$ the structure of a complex torus. Then (4) induces a Riemann form on $\mathfrak{a}$ by

$$\mathfrak{a} \times \mathfrak{a} \to \mathbb{Q}$$

$$(x, y) \mapsto \mathrm{Tr}_{E|\mathbb{Q}}(\overline{x} u y).$$

Realizing the CM classes

If we fix a CM order $\mathcal{O}$ in a CM *field* E and an isomorphism $\Phi : \mathcal{O} \otimes \mathbb{R} \xrightarrow{\sim} \mathbb{C}^g$, then the remaining data can be assembled into a groupoid $\mathcal{P}_E^-$ with objects

$$\mathcal{O}\text{-lattices in } E \text{ with a Riemann form.}$$

This is equivalent to the groupoid of rank 1 projective $\mathcal{O}$-modules with skew-Hermitian form valued in $\mathrm{Hom}_{\mathbb{Z}}(\mathcal{O}, \mathbb{Z})$. On a lattice which is an ideal of $\mathcal{O}$, the skew-Hermitian form is given by

$$(x, y) \mapsto [z \mapsto \mathrm{Tr}_{E|\mathbb{Q}}(\overline{x}uyz)].$$

Using the skew-Hermitian notation from previous talks, we have

$$\mathcal{P}_E^- \cong \mathcal{P}(\mathcal{O}, \mathrm{Hom}_{\mathbb{Z}}(\mathcal{O}, \mathbb{Z}), f \mapsto (x \mapsto -\overline{f(\overline{x})})$$

Realizing the CM classes

The CM theory then gives a functor

$$\mathrm{ST} : \mathcal{P}_E^- \to \mathcal{A}_g(\mathbb{C}).$$

The composite map with $\mathcal{A}_g(\mathbb{C}) \to \mathcal{SP}(\mathbb{Z})$:

$$\mathcal{P}_E^- \to \mathcal{SP}(\mathbb{Z})$$

is of particular interest. On the full subgroupoid of $\mathcal{O}$-lattices with integral-valued Riemann form, this simply realizes the lattice as a $\mathbb{Z}$-module and the Riemann form as a skew-symmetric form.

A lemma (for next week)

Specialize to $\mathcal{O} = \mathbb{Z}[\zeta_q]$, the ring of integers of $K_q = \mathbb{Q}(\zeta_q)$. Fix an isomorphism $\Phi : \mathcal{O} \otimes \mathbb{R} \xrightarrow{\sim} \mathbb{C}^g$.

Lemma

Let $\mathfrak{b}$ be a fractional ideal of $\mathcal{O}$. Then there exist lattices $\mathfrak{a}_1, \mathfrak{a}_2 \in \mathcal{P}^-_{K_q}$ such that

$$[\mathfrak{a}_1 \mathfrak{a}_2] = [\mathfrak{b}\overline{\mathfrak{b}}^{-1}]$$

in the ideal class group of $\mathcal{O}$.

This is used for...

Conclusion

Proposition (Next week)

Take $E = K_q$. The map

$$\mathcal{P}_E^- \to \mathcal{SP}(\mathbb{Z})$$

induces maps

$$\pi_{4k-2}^s(|\mathcal{P}_E^-|; \mathbb{Z}/q) \to \mathrm{KSp}_{4k-2}(\mathbb{Z}; \mathbb{Z}/q)$$

which are surjective for all $k \geq 1$.