F21 JUVITOP TALK: THE BV FORMALISM

NATALIA MARÍA PACHECO TALLAJ

As we say in Wyatt's talk, when we talk of a (Lagrangian) field theory, we mean the data of

(1) A spacetime manifold M
(2) A principal G-bundle E whose sections or connections are the space $\mathcal{F}$ of *fields*. G is called the *gauge group*.
(3) An action functional $S : \mathcal{F} \to \mathbb{R}$, which is the integral $\int_M \mathcal{L}(\phi)$ over spacetime of some Lagrangian density.
(4) Possibly some antifields.

The goal of a field theory is first to find the fields that extremize S, called the *physical fields* (or *critical locus* $\mathrm{Crit}(S)$), namely we want to solve the *Euler-Legrange equations* $\mathrm{d}S = 0$, and then to compute expected values of functions on the physical fields, called observables. One of the goals of these books and the seminar is understanding (some sheaf version of) the latter as a factorization algebra, but the goal of the talk today is understanding the geometric object on which they are defined $\mathrm{Crit}(S)$.

In physics, computations (say, of the critical locus) are often done perturbatively: given one solution ϕ_0 of the Euler-Lagrange equations, find nearby solutions by considering formal series expansions

$$\phi = \phi_0 + \epsilon\phi_1 + \epsilon^2\phi_2 + \dots$$

and solving Euler-Lagrange equations iteratively on the ϕ_n. Approximating to n-th order

$$\phi = \phi_0 + \epsilon\phi_1 + \epsilon^2\phi_2 + \dots + \epsilon^n\phi_n$$

the Euler-Lagrange equations become systems of simpler differential equations at each power of ϵ. As we saw in Ishan's talk, pointed FMPs provide a framework in which to compute deformations of the basepoint at finite order, as well as derived perturbations (ϵ in nonzero cohomological degree). The goal for today is to construct FMPs cut out by the Euler-Lagrange equations on a manifold. In fact, since we can also consider local solutions to the E-L equations on spacetime, we will actually consider sheaves of FMPs cut out by E-L equations.

0.1. **Getting started.** An equivalent way to express the 0-set $\mathrm{d}S = 0$ is as the intersection inside $T^*\mathcal{F}$ of Γ_{dS} and the 0-section $\mathcal{F}$. This is some geometric object whose functions look like

$$\mathcal{O}(\mathrm{Crit}(S)) = \mathcal{O}(\Gamma_{\mathrm{d}S}) \otimes_{\mathcal{O}(T^*\mathcal{F})} \mathcal{O}(\mathcal{F})$$

Date: September 2021.

If we, in addition, have a Gauge group, we have a large amount of degeneracy in this solution: fields that are gauge equivalent to each other have all the same physical properties, so to truly describe the "space of physical fields" we'd like to consider only one representative from each gauge orbit. We actually need to take the above intersection in the cotangent space of the quotient $\mathcal{F}/G$, a space whose functions look like

$$\mathcal{O}(\mathcal{F}/G) = \mathcal{O}(\mathcal{F})^G,$$

G-invariant functions on $\mathcal{F}$.

There are two problems

- If the action of G is not nice/free, $\mathcal{F}/G$ is highly singular, and
- even without considering the gauge group, if S is not nice, $\Gamma_{\mathrm{d}S} \cap \mathcal{F} \subseteq T^*\mathcal{F}$ is highly singular

The solution is computing these quotients and intersections in the realm of derived geometry. The *derived critical locus* should be the geometric object whose functions are, instead, a *derived tensor product*

$$\mathcal{O}(\mathrm{Crit}(S)) = \mathcal{O}(\Gamma_{\mathrm{d}S} \otimes_{\mathcal{O}(T^*(\mathcal{F}/G))} \mathcal{O}(\mathcal{F}/G)$$

and the derived quotient $\mathcal{F}/G$ should be the geometric object whose functions are the *derived invariants*

$$\mathcal{O}(\mathcal{F}/G) = \mathcal{O}(\mathcal{F})^G$$

Let's first compute the derived critical locus in finite dimensions.

1. Finite-dimensional BV formalism

Assume $\mathcal{F} = V$ is a finite dimensional vector space, and since we're working perturbatively, interpret V as $T_{x_0}V$ about some basepoint x_0 unless otherwise stated. $\mathcal{O}(V) = \widehat{\mathrm{Sym}}^* V^\vee$ (I guess we're using hats to get formal power series on V, perhaps because physics? unclear. add hat.)

Let's compute first, very explicitly, the case where the gauge group is trivial and $S = 0$, and then we will add more data retroactively to our computation. Then we want to compute the intersection of the two lagrangian submanifolds $\Gamma_{\mathrm{d}S} = V$ and V of T^*V. Supposing T^*V has spatial coordinates $q_1, \dots, q_n$ (the coordinates of V) and cotangent coordinates $p_1, \dots, p_n$, we see that, in coordinates

$$\begin{aligned}
\mathcal{O}(\mathrm{Crit}(S)) &= \mathcal{O}(V) \overset{\mathbb{L}}{\underset{\mathcal{O}(T^*V)}{\otimes}} \mathcal{O}(V) \\
&= \mathrm{Sym}(q_1, \dots, q_n) \overset{\mathbb{L}}{\underset{\mathrm{Sym}(q_i, p_i)}{\otimes}} \mathrm{Sym}(q_1, \dots, q_n) \\
&= \mathbb{C}[q_1, \dots, q_n] \overset{\mathbb{L}}{\underset{\mathbb{C}[q_1, \dots, q_n, p_1, \dots, p_n]}{\otimes}} \mathbb{C}[q_1, \dots, q_n, p_1, \dots, p_n]/(p_1, \dots, p_n)
\end{aligned}$$

To replace the RHS with a quasifree $\mathbb{C}[q_1, \dots, q_n, p_1, \dots, p_n]$-resolution, we take the full polynomial algebra in the q_i and p_i concentrated in degree 0, and kill the p_is in cohomology by adding degree -1 generators ξ_i whose differentials are the p_i:

$$\mathbb{C}[q_1, \dots, q_n, p_1, \dots, p_n, \xi_1, \dots, \xi_n],\ |\xi_i| = -1,\ d\xi_i = p_i$$

Taking the derived tensor product above then yields

$$\begin{aligned}&\mathbb{C}[q_1,\dots,q_n]\otimes_{\mathbb{C}[q_1,\dots,q_n,p_1,\dots,p_n]}\mathbb{C}[q_1,\dots,q_n,p_1,\dots,p_n,\xi_1,\dots,\xi_n]\\ \simeq\ &\mathbb{C}[q_1,\dots,q_n,\xi_1,\dots,\xi_n],\quad |\xi_i|=-1,\quad d\xi_i=0\end{aligned}$$

where $d\xi_1$ becomes 0 since p_i died. By inspection, we see this is $\mathcal{O}(T^*[-1]V)$: it's the function algebra we get by shifting the tangent coordinates in TV by 1.

If we add nonzero S then the p_i arent 0 anymore in $\mathcal{O}(\Gamma_{\mathrm{d}S})$ but rather $\partial_{q_i}S$, so $d\xi_i = p_i - \partial_{q_i}S$ in the quasifree resolution since instead of $p_i = 0$ we need to enforce $p_i = \partial_{q_i}S$ at the cohomology level, so the tensor product looks the same as an algebra but has nontrivial differential $d\xi_i = -\partial_{q_i}S$.

We also see that, in the $S = 0$ case, $\mathcal{O}(T^*[-1]V) = \mathcal{O}(V \oplus V^\vee[-1]) = \mathrm{Sym}(V[1]\oplus V)$ with 0 differential coming from the 0 differential in $V^\vee \to V$. If S is nonzero, $d\xi_i = \partial_{q_1}S$ gives $\mathcal{O}(T^*[-1]V)$ the differential $\vee \mathrm{d}S$.

$T^*[-1]V$ has a *shifted symplectic structure* because V pairs with $V^\vee$ (the symplectic form is $\omega = \sum \mathrm{d}g_i \wedge \mathrm{d}\xi_i$ with $\mathrm{d}g_i$ deg 1 and $\mathrm{d}\xi_i$ deg 0).

Now let's add a nontrivial group of gauge transformations. Then we want to compute $\mathrm{Crit}(S) = T^*[-1](V/\mathfrak{g})$, so we need to express the quotient $V/\mathfrak{g}$ in a derived way. As we said before, functions on the derived qutient are derived invariants of $\mathcal{O}(V)$. In Natalie's talk, we saw derived invariants are given precisely by the Chevalley-Eilenberg complex $C^*(\mathfrak{g}, V)$. We said $\mathcal{O}(V/\mathfrak{g})$ is the derived $\mathfrak{g}$-invariants of $\mathcal{O}(V)$. Lie algebra cohomology with coefficients in a $\mathfrak{g}$-module V is precisely the derived functors of $V \mapsto V^{\mathfrak{g}}$, so these derived invariants are given by the Chevalley-Eilenberg complex $C^*(\mathfrak{g}, \mathcal{O}(V))$ which we saw in Natalie's talk is $\mathcal{O}(\mathfrak{g}[1] \oplus V)$.
The derived quotient $\mathfrak{g}[1] \oplus V$ is in fact a dg manifold, with a differential

$$\mathfrak{g} \to V$$

encoding the $\mathfrak{g}$ action on V. This map, called the *infinitesimal gauge transformations*, is given by the derivative of

$$\begin{aligned}G &\to \mathcal{F}\\ g &\mapsto g x_0\end{aligned}$$

Then, the full critical locus is

$$\mathrm{Crit}(S) = \mathfrak{g}[1] \oplus V \oplus V^\vee[-1] \oplus \mathfrak{g}[-2]$$

This is called the *finite-dimensional classical BV complex.*
The important take-aways here are that the derived critical locus of a field theory is: a graded vector space concentrated in 2 degrees (fields and their dual) if there is no gauge group, and in 4 degrees (fields, infinitesimal gauge transformations, and their duals) if there is a gauge group, whose differential is constructed from the action functional, and that this whole structure is the total space of a shifted cotangent bundle, who has a -1-shifted symplectic structure.

[1] Then, since the index of each input to the symplectic pairing increased by 1, in L_∞-land we're actually looking at degree -3 pairings.
But spaces of fields are in fact not finite dimensional: they are sections of some bundle. To express them, Costello and Gwilliam use the formalism of FMPs and L_∞-algebras.

2. Recollections on Maurer-Cartan equations and the FMP $B\mathfrak{g}$

Recall from Natalie's talk that an L_∞ algebra over a field k was a $\mathbb{Z}$-graded k-vector space $\mathfrak{g}$ equipped with a sequence of multilinear maps

$$\ell_n : \mathfrak{g}^{\otimes n} \to \mathfrak{g}$$

of cohomological degree $2-n$ satisfying graded andtisymmetry and homotopy Jacobi identities, and that the 1-Jacobi rule told us ℓ_1 is a differential on $\mathfrak{g}$.
Recall from Ishan's talk that a pointed formal moduli problem is a functor

$$F : \mathrm{CAlg}_{\mathrm{art}} \to \mathrm{sSet}$$

satisfying pointedness $F(k) \simeq *$ and a pullback condition (see Ishan's notes).
In Ishan's talk we furthermore saw that given an L_∞ algebra $\mathfrak{g}$ and an Artinian dg k-algebra $(R, \mathfrak{m}_R)$, there is a Maurer-Cartan equation over $\mathfrak{g} \otimes \mathfrak{m}_R \otimes \Omega^*(\Delta^n)$ given by

$$\mathrm{d}\alpha + \sum_{n \geq 2} \frac{1}{n!} \ell_n(\alpha, \ldots, \alpha) = 0$$

and if we let

$$\mathrm{MC}(\mathfrak{g} \otimes \mathfrak{m}_R)$$

denote the simplicial set of elements satisfying this MC equation, then

$$B\mathfrak{g} : (R, \mathfrak{m}_R) \mapsto \mathrm{MC}(\mathfrak{g} \otimes \mathfrak{m}_R)$$

is a pointed formal moduli problem called $B\mathfrak{g}$.

What exactly are we looking for? Recall the goal of the BV formalism as presented in CG is to, given a critical point ϕ_0 of the action functional, find other roots of dS near it by describing its formal neighborhood. A formal moduli problem F describes precisely a formal neighborhood of its basepoint (consider $F(k[\epsilon]/\epsilon^n$ as describing nth order approximations to the basepoint), and the FMP $B\mathfrak{g}$ describes precisely the root set of a differential equation. Therefore, in order to locally describe the derived critical locus of a certain field theory in this formalism, *more or less* all we need to do is find a Lie algebra whose Maurer-Cartan equations are the equations of motion. [2]

[1]The complex we wrote out above is in fact the dg manifold representing the derived critical locus, which is analogous to the FMP representing the DCL in infinite dimensional case. We'll see later that L_∞ algebras correspond to right-shifted tangent space of their FMPs, so to get the analog of the BV L_∞ algebra from this, we should shift right by one.
[2]Emphasis on more or less. In fact the Maurer-Cartan equation must (and will) encode not only the EOM for the fields in the theory that we'd get from looking at the variation in S classically, but also the EOM of the antifields, the Lie bracket on the space of infinitesimal gauge symmetries, and the way this Lie algebra acts on the original space of fields.

We witnessed in the classical case that $d\mathrm{Crit}(S)$ has a -1-shifted symplectic structure.

Lemma 2.1. *For $\mathfrak{g}$ a finite-dimensional L_∞-algebra, a k-shifted symplectic structure on $B\mathfrak{g}$, which is a k-shifted pairing on $T_pB\mathfrak{g}$, is the same data as a $k-2$-shifted pairing on $\mathfrak{g}$.*

Sketch/idea. $\mathcal{O}(B\mathfrak{g}) = C^*(\mathfrak{g}) = \widehat{\mathrm{Sym}}(\mathfrak{g}^\vee[1])$. Under this correspondence,

$$\mathfrak{g}\text{-module } \mathfrak{g}[1] \leftrightarrow \ TB\mathfrak{g}$$

so we should think of

$$\mathfrak{g} \simeq T_pB\mathfrak{g}[-1]$$

Furthermore, since we can consider local solutions of the EOM, we actually want to work with sheaves of L_∞ algebras/FMPs.

Definition 2.2. A *local L_∞ algebra* on a manifold M is the data of

- a graded vector bundle L on M whose space of smooth sections is denoted $\mathcal{L}$
- a differential operator $d : \mathcal{L} \to \mathcal{L}$ of cohomological degree 1 and square 0
- a collection of polydifferential operators $\ell_n : \mathcal{L}^{\otimes} n \to \mathcal{L}$ for $n \geq 2$ which are alternating, of cohomological degree $2-n$, and endow $\mathcal{L}$ with the structure of an L_∞ algebra

Such an object yields a homotopy sheaf (satisfies Cech descent) of FMPs that assigns to $U \subset M, (R, \mathfrak{m}_R) \in \mathrm{CAlg}^{\mathrm{Art}}$ the sSet

$$B\mathcal{L}(U)(R) = \mathrm{MC}(\mathcal{L}(U) \otimes \mathfrak{m}_R)$$

of MC elements of the L_∞ algebra $\mathcal{L}(U)$ with coefficients in $\mathfrak{m}_R$.

In this setting, the correct notion of dual is a density-valued dual

$$\mathcal{L}^! = \mathcal{L}^\vee \otimes \mathrm{Dens}_M$$

and a degree k pairing is a symmetric map $L \otimes L \to \mathrm{Dens}(M)[k]$

3. The Classical BV algorithm

Given a classical field theory as defined at the beginning of this talk, we describe an algorithm to construct the L_∞ algebra $\mathcal{L}$ such that $B\mathcal{L}$ is the derived critical locus of the field theory.

Step 1. We start with the so-called BRST fields $\mathscr{L}$, a local L_∞-algebra concentrated in degrees 0 and 1 defined as follows: In degree 1 we have the original space of fields (sections or connections on the relevant bundle) *near* ϕ_0, meaning take "tangent spaces" at ϕ_0. [3] In degree 0 we have the Lie algebra of the group of gauge transformations (automorphisms of the bundle fixing the base). The map $\mathscr{G}_0 \to \mathscr{L}_1$ is defined in the same way as in the finite dimensional case: the derivative of the act-on-ϕ_0 map (here, we can define it using the Lie group exponential). The

[3] In the sense that the space of sections or connections on a bundle is affine modelled on sections or forms on that bundle, so we have a notion of "tangent space near" ϕ_0.

infinitesimal gauge transformations $\mathscr{G}_0$ are usually considered, by phycists, additional fields (called *ghosts*) that generate the gauge symmetries.
$\mathscr{G}_0$ has a Lie bracket, and $\mathscr{G}_0$ acts on $\mathscr{L}_1$ by acting on the coefficients.[4] Together, these two things give us a nontrivial ℓ_2 on the BRST fields (which has degree $k-2=2-2=0$).

$$\text{Lie bracket} \circlearrowleft \mathscr{G}_0 \underset{\text{action}}{\overset{d}{\rightrightarrows}} \mathscr{L}_1$$

Step 2. Next we add the anti-fields $\mathscr{L}^![-3]$ to get a new local Lie algebra

$$\mathscr{L} \oplus \mathscr{L}^![-3] = \mathscr{L}_0 \to \mathscr{L}_1 \quad \mathscr{L}_2^! \to \mathscr{G}_3^!$$

where, because we took a -3 shift, we have that elements of $\mathscr{L}_1$ pair with elements of $\mathscr{L}_2^!$ to give things in $\mathrm{Dens}(M)[-3]$, so we are poised to have a 3-shifted pairing on the complex we are constructing as expected from the previous analysis.

Step 3. To add the differential and higher brackets, express your action functional S by the taylor expansion

$$S(\phi) = \sum_{k\geq 2} \frac{1}{k!} \int_M \langle \ell_{k-1}(\phi^{\otimes k-1}), \phi \rangle$$

with $\ell_j : \mathscr{L}^{\otimes j} \to \mathscr{L}^![-j-2]$ and $\langle \ , \ \rangle : \mathscr{L} \otimes \mathscr{L}^! \to \mathrm{Dens}_M$. [5] Let ℓ_1 be the differential $\mathscr{L}_1 \to \mathscr{L}_2^!$, and let ℓ_k be the higher brackets of

$$\mathscr{G} \to \mathscr{L} \xrightarrow{\ell_1} \mathscr{L}^! \to \mathscr{G}^1$$

This is the classical BV complex.

Note that this construction is as advertised: the MC equation at the base field $\mathbb{R}$ recovers the Euler-Lagrange equations for the fields and antifields (and other data related to the gauge action).

3.1. **A note about basepoints.** You may have noticed that as it stands, the basepoint of the EFMP we are constructing is always 0:

$$B\mathfrak{g}(k) = \mathrm{MC}(\mathfrak{g} \otimes \mathfrak{m}_k) = \mathrm{MC}(\mathfrak{g} \otimes 0)$$

and you might be worried: how much can we really say about qft if we can only work infinitesimally close to the 0 field.
You actually can compute formal neighborhoods of nontrivial fields in this formalism, but it requires what is called a *curved L_∞ algebra*, which is an L_∞-algebra with an added 0-ary operation

$$\ell_0 : \mathfrak{g}^{\otimes 0} \simeq k \to \mathfrak{g}$$

[4]Our gauge theory is happening over some principal G-bundle $P \to M$, whose automorphisms are $\Omega^0(M, \mathrm{ad}P)$, and even if the fields are sections of some associated bundle, there is a natural G-action on the fibre so we can perform this construction with the relevant Lie algebra action.
[5]Notice than in field theories, the Lagrangians always start with kinetic terms, but there is also a purely mathematical reason which is that if S had subquadratic terms, the basepoint of our FMP would not be a solution to $\mathrm{d}S = 0$.

$\ell_0(1)$ is called the *curvature* of the L_∞-algebra, and the 1-Jacobi identity no longer gives us $\ell_1^2 = 0$, but rather

$$\ell_1^2 = \pm \ell_2(\ell_0, -)$$