Symmetries of Field Theories

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1 Introduction

In this talk, we review Noether's theorem in Hamiltonian mechanics (symplectic geometry) and motivate Noether's theorem in classical field theory. Then we state and sketched a proof of Noether's theorem in classical field theory via Costello-Gwilliam's BV formalism.

2 Noether's Theorem in Hamiltonian Mechanics

The slogan we always heard about Noether's theorem is the following: "continuous symmetry gives conserved quantities". We first review how this works in Hamiltonian mechanics and discuss some less-discussed part about Noether's theorem.

In Hamiltonian mechanics, we have a phase space (X, ω), which is a symplectic manifold (normally the cotangent space of the configuration space), and the energy function $H : X \to \mathbb{R}$. Let $SympVect(X)$ be the subspace of symplectic vector fields, that is, vector fields on X that preserves ω. The observables $\mathcal{O}(X)$ is the space of functions on X. They form a Poisson algebra due to ω.

Continuous symmetry means actions of lie algebra. Let $\mathfrak{g}$ be a lie algebra.

Definition 2.1. An action of $\mathfrak{g}$ on (X, ω, H) is

1. a map of lie algebra $\mathfrak{g} \to SympVect(X)$. That is, each element of $\mathfrak{g}$ gives a infinitesimal action that preserves the symplectic structure.

2. The image of $\mathfrak{g}$ preserves H. That is, the energy H is also preserved.

Noether's theorem says that we should get conserved quantities, since conserved quantities are really just functions, we expect a map $\mathfrak{g} \to \mathcal{O}(X)$.

How can we construct such a map? Notice that $SympVect(X) \simeq \Omega^1_{cl}(X)$ the space of closed vector fields on X by contracting with ω. Assuming that $H^1(X, \mathbb{R}) = 0$, then $\Omega^1_{cl}(X) = \Omega^1_{ex}(X)$, all closed forms are exact.

Note that we have a short exact sequence

$$0 \to \mathbb{R} = H^0(X; \mathbb{R}) \to \mathcal{O}(X) \to \Omega^1_{ex}(X) \to 0 \tag{2.2}$$

Thus we are trying to lift a map $\mathfrak{g} \to \Omega^1_{ex}(X)$ to $\mathfrak{g} \to \mathcal{O}(X)$. Note that this isn't always possible, there is an obstruction in $H^1(\mathfrak{g}; \mathbb{R})$. However, an central extension $\tilde{\mathfrak{g}}$ of $\mathfrak{g}$ lifts, where $\tilde{\mathfrak{g}}$ is simply the pullback:

$$\begin{tikzcd} \tilde{\mathfrak{g}} \arrow[r] \arrow[d] & \mathfrak{g} \arrow[d] \\ \mathcal{O}(X) \arrow[r] & \Omega^1_{ex}(X). \end{tikzcd} \tag{2.3}$$

We see that we get conserved quantities, at the cost of doing a central extension.

Remark 2.4. Actually, getting a central extension is a good thing! The obstruction in $H^1(\mathfrak{g}; \mathbb{R})$, should not be viewed as an obstruction to lift, but rather invariants of the system. Think about Virasoro algebra (central charge) and Kac-Moody algebras (the level). Physically, these invariants help us distinguished systems (and their phases).

It seems like we got our Noether's theorem, however, there is more structure than that: since $\mathfrak{g}$ acts on the system, it acts on the space of observables $\mathcal{O}(X)$. Recall that $\mathcal{O}(X)$ has a Poisson structure and acts on itself. The real meat of the theorem is the compatibility of the map $\tilde{\mathfrak{g}} \to \mathcal{O}(X)$ and the two actions:

Theorem 2.5. *The action of $\mathfrak{g}$ on $\mathcal{O}(X)$ agrees with the action of $\mathfrak{g}$ through the map $\tilde{g} \to \mathcal{O}(X)$.*

Note that the central $\mathbb{R}$ of $\tilde{g}$ is from the constant functions on X, and they acts trivial with repsect to the Poisson structure, thus the action on $\tilde{\mathfrak{g}}$ on $\mathcal{O}(X)$ factors through $\mathfrak{g}$.

This also tells us how we should also think about Noether's theorem: the action of $\mathfrak{g}$ on the system can be expressed in terms of the (local) observables of the system.

3 Noether's theorem in BV formalism

Now that we understand what Noether's theorem really is, let's go to classical field theory in BV's setting. Recall that a classical field theory on spacetime manifold X is a (local) L_∞ algebra $\mathcal{M}$ with a -3-shifted invariant pairing. Our continuous symmetry is going to be an L_∞ algebra $\mathcal{L}$.

First we have to figure out what does it mean for $\mathcal{L}$ to act on the classical field theory $\mathcal{M}$. Ignore the pairing for now, what does it mean for an L_∞ algebra to act on another?

Well, an action on $\mathcal{L}$ on $\mathcal{M}$ should be a map $\mathcal{L} \to Der(\mathcal{M})$, where $Der(\mathcal{M})$ is the L_∞ algebra of derivations (think about ordinary lie algebras). Equivalently, this is equivalent to giving a semi-direct product structure on $\mathcal{L} \oplus \mathcal{M}$, that is, a L_∞ structre on $\mathcal{L} \oplus \mathcal{M}$, which we will write as $\mathcal{L} \ltimes \mathcal{M}$, together with a short exact sequence of L_∞ algebras:

$$0 \to \mathcal{L} \to \mathcal{L} \ltimes \mathcal{M} \to \mathcal{M} \to 0 \tag{3.1}$$

We will take this as our definition.

Now we only need to define what does it mean for an action to be compatible with the invariant pairing:

Definition 3.2. An action on $\mathcal{L}$ on $\mathcal{M}$ preserves the pairing if for every $\alpha_1, ..., \alpha_i \in \mathcal{L}$ and $\beta_1, ..., \beta_j \in \mathcal{M}$, we have that

$$< l_{i+j-1}(\alpha_1 ... \alpha_r ... \beta_1 ... \beta_{j-1}), \beta_j > \tag{3.3}$$

is totally anti-symmetric under permutation of β_j.

Note that $(\alpha_1, ...)$ means the action of α_i on the β.

There are equivalent definitions:

Lemma 3.4. *The following are equivalent:*

1. *an L_∞ action of $\mathcal{L}$ on $\mathcal{M}$.*
2. *An element $S^{\mathcal{L}} \in Act(\mathcal{L}, \mathcal{M})$ satisfying the Mauer-Cartan equation:*
$$(d_{\mathcal{L}} + d_{\mathcal{M}})S^{\mathcal{L}} + 1/2\{S^{\mathcal{L}}, S^{\mathcal{L}}\} = 0 \tag{3.5}$$
.
3. *The sum $S^{tot} = S + S^{\mathcal{L}} \in C^*_{red,loc}(\mathcal{L} \oplus \mathcal{M})/C^*_{red,loc}(\mathcal{L})$ satisfying the classical master equation:*
$$d_{\mathcal{L}} S^{tot} + 1/2\{S^{\mathcal{L}}, S^{\mathcal{L}}\} = 0. \tag{3.6}$$
$S \in C^*_{red,loc}(\mathcal{M})$ is the action functional associated to $\mathcal{M}$.

The third condition tells us that an action of $\mathcal{L}$ on $\mathcal{M}$ is equivalent to extending $\mathcal{M}$ to a sheaf of classical field theory over the formal moduli problem $B\mathcal{L}$.

The fact that S^{tot} only satisfies the classical master equation up to $C^*_{red,loc}(\mathcal{L})$ tells us that there is an $\alpha \in C^*_{red,loc}(\mathcal{L})$ with

$$d_{\mathcal{L}} S^{tot} + 1/2\{S^{\mathcal{L}}, S^{\mathcal{L}}\} = \alpha \otimes 1 \tag{3.7}$$

in $C^*_{red,loc}(\mathcal{L} \oplus \mathcal{M})$. This gives us an obstruction class $\mathfrak{o} \in H^1(C^*_{red,loc}(\mathcal{L}))$.

Now we are ready to state the Noether's theorem:

Theorem 3.8. *If $\mathcal{L}$ acts on $\mathcal{M}$, there exists an central extension $\tilde{\mathcal{L}}$ of $\mathcal{L}$ (by $\mathbb{R}[-1]$) with a map of P_0 algebras*

$$C^*_{red,loc}(\tilde{\mathcal{L}}) \to Obs(\mathcal{M}) \tag{3.9}$$

such that the action of $\mathcal{L}$ on $Obs(\mathcal{M})$ agrees with this map (up to homotopy).

Remark 3.10. The lift $\tilde{\mathcal{L}}$ is the one corresponding to the obstruction class $\mathfrak{o} \in H^1(C^*_{red,loc}(\mathcal{L}))$. The fact that the central $\mathbb{R}$ is in degree -1 is because everything here is -1-shifted (as oppose to the 0-shifted case of Hamiltonian mechanics).

Proof. The idea is that S^{tot} twisted by α on $C^*_{red,loc}(\tilde{\mathcal{L}} \oplus \mathcal{M})$ satisfies the classical master equation, and that precisely encodes a map of L_∞ algebras $\mathcal{L} \to \mathcal{M}$. □