QUANTIZATION OF (FREE) FIELD THEORIES

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1. Introduction

In this talk, we will discuss quantization of field theories à la Costello–Gwilliam, via the machinery of BV quantization and factorization algebras. There are many mathematical incarnations of the physical idea of quantizing a classical field theory, and the one that Costello–Gwilliam addresses is the *deformation quantization of the Poisson algebra of observables.*

We will start by briefly reviewing the physical idea of quantization, where the quantum deformation parameter $\hbar$ appears naturally. Next, we will discuss the operadic formalism used in Costello–Gwilliam to approach the quantization problem. Finally, we will see an example of this formalism being applied to quantization of a free field theory, as promised by the title of the talk.

2. Physics preliminaries: Quantization

This section is largely for physical motivation, and can be skipped for people who are only interested in Costello–Gwilliam's treatment of the problem. Moreover, I will not provide clear mathematical definitions of the concepts involved.

We will very briefly review the physical idea of quantization. There are largely two approaches to describing a quantization of a classical field theory. One is via canonical quantization, which is closer to the Hamiltonian formalism of field theory. The other is via path integrals, which is closer to the Lagrangian formalism of field theory.

In the perturbative setting, the preferred approach is via path integrals. Recall that the data of a classical field theory consists of

- A spacetime M,
- Fields ϕ, which are formal variables taking values in sections of some fiber bundles, and the corresponding space of field configurations $\mathcal{F}$,
- and the action functional
$$S = S(\phi) = \int_M \mathcal{L}(\phi, \partial_\mu \phi, \dots)$$
which is a integral over the spacetime of the Lagrangian density $\mathcal{L}$, which is in turn some differential polynomial in ϕ.

Then one proceeds by variational calculus to compute the stationary points of S, which are characterized as the solutions of an elliptic PDE (called the equation of motion, or Euler–Lagrange equation). The path integral approach roughly entails

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that we put some (mathematically ill-defined) measure $\mathcal{D}\phi$ on the space $\mathcal{F}$ of field configurations and consider the following "oscillatory integral"

$$Z = \int_{\mathcal{F}} \mathcal{D}\phi \exp\left(\frac{i}{\hbar} S(\phi)\right) = \int_{\mathcal{F}} \mathcal{D}\phi \exp\left(\frac{i}{\hbar}\int_M d\mathbf{x}\ \mathcal{L}(\phi)\right)$$

called the partition function.

The asymptotic analysis of oscillatory integrals, known as the *stationary phase approximation*, formally tells us that this integral will have a dominant contribution from the stationary points of $S(\phi)$ (the classical contribution), and some corrections (in positive orders of $\hbar$) (the quantum contributions). There is a diagrammatic scheme, called *Feynman diagrams*, which allows one to enumerate these contributions in terms of some graphs. The tree graphs correspond to the classical contributions, and graphs with cycles (loops) correspond to the quantum contributions.

Remark 2.1. The reduction in terms of Feynman diagrams is a version of a perturbative series. By focusing on the critical points of the action functional and the perturbative expansion in the vicinity of those critical points, we completely ignore the non-perturbative effects (such as instantons and solitons), unless we introduce some asymptotic summation methods.

The integrals over the tree diagrams are often convergent, but the integrals over the loop diagrams are very likely divergent. Physicists developed a way to deal with this divergence known as *renormalization*, which was made mathematically rigorous by the work of Costello.

Remark 2.2. For quadratic Lagrangians (i.e. free theories), there are no loop terms in the $\hbar$-perturbative expansion (in fact there are no trees with internal vertices either), and hence the issues of renormalization are invisible in this case. This property makes their quantization much easier.

The path integral approach is preferred by physicists because they immediately give definitions and computation methods for correlation functions (expectations of observables) which can be tested against experiments.

One can also consider the algebraic structures of the observables after quantization. In this context, the preferred language is via Hamiltonian formalism. Mathematically, we are looking at

- The phase space, which is the space of classical (i.e. physical) field configurations. This may be interpreted as the (derived) critical locus
$$\mathrm{Crit}(S) = \Gamma_{dS} \cap \underline{0}_{\mathcal{F}}$$
in $T^*\mathcal{F}$.
- A (shifted) symplectic structure on the phase space, and the induced (shifted) Poisson algebra structure on the algebra of functions on the phase space.

Remark 2.3 (Comparison with mechanics). More often, the phase space is described as the cotangent bundle of the configuration space in the context of mechanics. The field theory describing mechanics of a bosonic particle on a Riemannian manifold (M, g) is the $d = 1$ (worldline) theory with $\phi : \mathbb{R} \to M$ as the fields (sections of the nonlinear bundle $\mathbb{R} \times M \to \mathbb{R}$). The classical phase space for this theory can be parametrized by the initial condition of the physical configurations ϕ_{phys} (for free theories with no potential, these are the parametrized geodesics). The initial conditions are naturally identified with the cotangent bundle T^*M of the underlying manifold M.

It is shown in [CG16a](Section 2.4) that indeed $\mathcal{O}(\mathrm{Crit}S)$ is quasi-isomorphic to $\mathcal{O}(T^*M)$, at least explicitly for $M = \mathbb{R}$. In other words, the scalar field theory on $\mathbb{R}$ is identified with the worldline theory of a particle on a line.

To quantize, one replaces the phase space with a Hilbert space $\mathcal{H} = L^2(\mathcal{F})$ that describes the half-densities on the space of all field configurations. The elements of $\mathcal{H}$ (up to squaring and normalizing) carry the interpretation as probability density functions over $\mathcal{F}$. The observables (functions on classical phase space) are replaced by (Hermitian) operators on $\mathcal{H}$.

The space of operators on $\mathcal{H}$ carry the usual commutator bracket. If we denote the quantization of the classical observable A by $\hat{A}$, the bracket is expected to be related to the classical Poisson bracket by

$$[\hat{A}, \hat{B}] = i\hbar\{A, B\} + o(\hbar).$$

Hence, after quantization, the algebra of observables becomes an associative algebra whose commutator bracket is a $\hbar$-multiple of the Poisson bracket on the algebra of classical observables, up to higher order corrections in $\hbar$.

A strictly weaker algebraic approach to quantization then concerns a $\hbar$-deformation of a Poisson algebra A over k to an associative algebra $A_\hbar$ over $k[\![\hbar]\!]$, such that the brackets satisfy the relation above. This algebraic problem is commonly referred to as *deformation quantization* of quantum mechanics.

Remark 2.4. For canonically conjugate coordinates $q, p = \partial\mathcal{L}/\partial\dot{q}$ in classical mechanics, the corresponding quantized operators form a Heisenberg Lie algebra together with $\hbar$ considered as a central element:

$$\{q, p\} = 1 \implies [\hat{q}, \hat{p}] = i\hbar.$$

The Costello–Gwilliam approach generalizes this picture of quantum mechanics to quantum field theory by considering

- The BV–BRST description of the classical phase space of a (gauged) field theory via derived critical locus,
- The P_0 factorization algebra structure on the classical observables $\mathrm{Obs}^{\mathrm{cl}}$,
- The BD_0 factorization algebra structure on the quantum observables Obs^q, arising from a deformation of P_0 structure on $\mathrm{Obs}^{\mathrm{cl}}$.

Applied to the $d = 1$ worldline theory of a particle in a linear space, this recovers the usual deformation quantization for quantum mechanics. The key property for this application is that Obs^q for spacetime $\mathbb{R}$ (the wordline) is a locally constant factorization algebra, hence defines a E_1-algebra. [CG16a](Lemma 3.0.1)

3. Quantization of observables à la Costello–Gwilliam

We now discuss the quantization of observables following Costello–Gwilliam. As mentioned in previous sections, the formalism of Costello–Gwilliam addresses the deformation quantization problem for the Poisson algebra of observables.

We first describe the algebraic structure one puts on classical observables, namely the P_0 factorization algebra structure. Then we will describe the algebraic problem of quantizing these, formulated operadically using BD_0-algebras.

3.1. **The algebraic structure on classical observables of a field theory.** We first recall the algebraic structure on classical observables. Let M be our spacetime and $\mathcal{L}$ be the elliptic L_∞ algebra of a classical field theory.

Definition 3.1. The **classical observables with support in** U is the cdga

$$\mathrm{Obs}^{\mathrm{cl}}(U) = C^*(\mathcal{L}(U)),$$

where C^* denotes the Chevalley–Eilenberg cochains of an L_∞ algebra. The *factorization algebra of classical observables* $\mathrm{Obs}^{\mathrm{cl}}$ assigns $\mathrm{Obs}^{\mathrm{cl}}(U)$ to U.

The motivation for this definition is that $\mathcal{L}$ corresponds to the FMP associated to the Euler–Lagrange equations of the theory. By Ishan's talk, the Chevalley–Eilenberg cochains of a L_∞ algebra are identified with the functions on the corresponding FMP.

It will be more convenient for us to write this definition in terms of the fields of the theory. This amounts to replacing $\mathcal{L}$ with

$$\mathcal{E} = \mathcal{L}[1] = T_{\phi_0} B\mathcal{L}.$$

Recall that $\mathcal{L}$ is a classical field theory in the sense that it is a local elliptic L_∞ algebra equipped with a (-3)-shifted symmetric pairing $L \cong L^![-3]$ for the underlying graded vector bundle for $\mathcal{L}$. The data of the action functional define the coherent differentials for $\mathcal{L}$.

For $\mathcal{E}$, the pairing on the underlying vector bundle

$$E \cong E^![-1] \iff E \otimes E \to \mathrm{Dens}[-1]$$

is now (-1)-shifted and antisymmetric, defining the (-1)-shifted symplectic structure on E. This symplectic structure naturally induces a $(+1)$-shifted Poisson bracket $\{-,-\}$ on the functions on $E^\vee$, and moreover on all local functionals on E ([CG16b] Section 4.5 for precise definition). The action functional S must satisfy the *classical master equation*

$$\{S, S\} = 0$$

with respect to this bracket, which is equivalent data to the L_∞ relations on $\mathcal{E}[-1] = \mathcal{L}$. After this shift, we can rewrite the definition as follows.

Definition 3.2. The **classical observables with support in** U is the cdga

$$\mathrm{Obs}^{\mathrm{cl}}(U) = (\mathcal{O}(\mathcal{E}(U)), \{S, -\}).$$

Now we are ready to describe the precise algebraic structure one can put on $\mathrm{Obs}^{\mathrm{cl}}$.

Definition 3.3. A factorization algebra $\mathcal{F}$ is a **1-shifted Poisson factorization algebra** if

- Each $\mathcal{F}(U)$ is a P_0-algebra,
- The corestriction maps are maps of P_0-algebras,
- The sections from disjoint open subsets U_i under the factorization algebra maps

  $$\mathcal{F}(U_1) \otimes \cdots \otimes \mathcal{F}(U_n) \to \mathcal{F}(V)$$

  Poisson commute in $\mathcal{F}(V)$.

Remark 3.4. There is a general notion of structured factorization algebras for Hopf operads, of which this definition is a special case. [CG16b], Section 2

Proposition 3.5. *The classical observables* $\mathrm{Obs}^{\mathrm{cl}}$ *has the structure of a 1-shifted Poisson factorization algebra.*

Proof. We will treat the **free field** case. In the free field situation, we have a simpler description of the classical observables:

$$\mathrm{Obs}^{\mathrm{cl}}(U) = \mathcal{O}(\mathcal{E}(U)) = \widehat{\mathrm{Sym}}(\mathcal{E}^\vee(U)),$$

where the dg structure on the symmetric algebra is simply the differential on $\mathcal{E}^\vee$ extended as a derivation. In particular, to define a Poisson bracket structure on $\mathrm{Obs}^{\mathrm{cl}}(U)$, it suffices to define the pairing on $\mathcal{E}^\vee(U)$ and extend by Leibniz rule.

Now we identify, using the (-1)-shifted symplectic structure,

$$\mathcal{E}(U)^\vee = \overline{\mathcal{E}}^!_c(U) \cong \overline{\mathcal{E}}_c(U)[1]$$

where $\overline{\mathcal{E}}_c$ denotes the compactly supported distributional sections of E.

Recall that on $\mathcal{E}_c(U)$ we have a natural (-1)-shifted antisymmetric pairing arising from the fiberwise pairing on the graded vector bundle E:

$$\langle \cdot, \cdot \rangle : \mathcal{E}_c(U) \otimes \mathcal{E}_c(U) \to \mathrm{Dens}[-1] \xrightarrow{\int_M} \mathbb{R}[-1],$$
$$\langle \phi, \psi \rangle = \int_{x \in M} \langle \phi(x), \psi(x) \rangle_{\mathrm{loc}}.$$

After shifting, we obtain a $(+1)$-shifted antisymmetric pairing

$$\{-,-\} : \mathcal{E}_c(U)[1] \otimes \mathcal{E}_c(U)[1] \to \mathrm{Dens}[1] \to \mathbb{R}[1].$$

Hence, *ignoring the functional analytic issue* of comparing the distributional sections of E with the smooth sections of E, this completes the construction of the $(+1)$-shifted Poisson bracket.

It remains to check that this Poisson bracket is compatible with the factorization algebra maps, that is, the observables supported in disjoint opens must Poisson-commute. This follows from the definition of our pairing in terms of integral over M of the local pairing. □

Remark 3.6. In practice, due to the functional analytic issue that we glossed over, it is not possible to define a P_0 structure on $\mathrm{Obs}^{\mathrm{cl}}(U)$, but in fact we can only define it for the subalgebra of *smooth* observables $\widetilde{\mathrm{Obs}}^{\mathrm{cl}}(U) = \mathcal{O}(\overline{\mathcal{E}}(U))$. The inclusion

$$\widetilde{\mathrm{Obs}}^{\mathrm{cl}}(U) \to \mathrm{Obs}^{\mathrm{cl}}(U)$$

is still a quasi-isomorphism.

3.2. **Operadic formulation of the deformation quantization problem.** As in deformation quantization, where the Poisson operad interpolates the commutative operad and the E_1 operad (i.e., "Poisson brackets want to be quantized"), a 1-shifted Poisson structure admits a natural notion of deformation quantization, which we will describe now.

Definition 3.7. The P_0 **operad** is the graded operad generated by a commutative associate product $- * -$ of degree 0 and a Poisson bracket $\{-,-\}$ of degree 1.

Definition 3.8. The E_0 **operad** is the operad with $E_0(0) = \mathbb{R}$ and $E_0(n) = 0$ for $n > 0$.

That is, a E_0-algebra in a symmetric monoidal category $\mathcal{C}$ is just an object A with a distinguished morphism $1_{\mathcal{C}} \to A$.

As P_n-algebras (with $(1-n)$-shifted Poisson brackets) arise naturally as semiclassical (homology) limits of E_n-algebras, P_0 algebras are posing to be semiclassical limits of E_0 algebras.

Definition 3.9. The **Beilinson–Drinfeld (BD) operad** is the dg operad over $\mathbb{R}[\![\hbar]\!]$ is

$$BD = P_0 \otimes_{\mathbb{R}} \mathbb{R}[\![\hbar]\!]$$

equipped with the differential

$$d(- \star -) = \hbar\{-,-\}$$

on $BD(2)$.

In particular, for a flat dg $\mathbb{R}[\![\hbar]\!]$-module M, a BD algebra structure on M amounts to a

- A commutative associative product $- \star -$ of degree 0,
- A Poisson bracket $\{-,-\}$ of degree 1, internal differential $d = d_M$ being a derivation for this bracket,
- and the relation

  $$d(m \star n) = (dm) \star n + (-1)^{|m|} m \star (dn) + (-1)^{|m|} \hbar\{m,n\}.$$

  being satisfied.

This is a natural operad that interpolates between P_0 and E_0, in the following sense:

Lemma 3.10. *There is an isomorphism of operad and a quasi-isomorphism of operads over* $\mathbb{R}(\!(\hbar)\!)$*,*

$$BD \otimes_{\mathbb{R}[\![\hbar]\!]} \mathbb{R} \cong P_0, \quad BD \otimes_{\mathbb{R}[\![\hbar]\!]} \mathbb{R}(\!(\hbar)\!) \simeq E_0 \otimes \mathbb{R}(\!(\hbar)\!),$$

respectively.

Now a deformation quantization problem for the classical observables admits the following interpretation as a lifting problem:

Definition 3.11. A **BD quantization** of a P_0 factorization algebra $\mathcal{F}$ is a BD factorization algebra $\mathcal{F}_\hbar$ and a quasi-isomorphism

$$\mathcal{F}_\hbar \otimes_{\mathbb{R}[\![\hbar]\!]} \mathbb{R} \simeq \mathcal{F}$$

of P_0 factorization algebras.

Now we can state the theorem of Costello–Gwilliam.

Theorem 3.12 (Costello–Gwilliam)**.** *Given a classical* ***free*** *field theory, its* P_0 *factorization algebra of classical observables* $\mathrm{Obs}^{\mathrm{cl}}$ *admits a BD quantization* Obs^q*.*

We will prove this theorem in the next section.

Remark 3.13. There is a weaker notion of quantization of P_0-algebras, which asks for a

- E_0-algebra over $k[\![\hbar]\!]$ such that modulo $\hbar$ recovers the original P_0-algebra with

- Correspondence (at a cohomological level) between the Poisson bracket and the bracket measuring the failure to lift the product structure to the E_0 algebra.

At this level, the quantization theorem is completely proven for all classical field theories (with a choice of BV quantization), while the full BD quantization theorem is only proved for free theories (for more details, see [CG16b] Section 2.3 and 2.4). Note that the data of a BV quantization is invisible in the free field case, as there is no issue of renormalization.

4. Example: Quantizing a free scalar field theory

A free field theory is a field theory with no interaction. In the Lagrangian description of a field theory, such a theory is characterized by having a Lagrangian density that is quadratic in the field variables. That is, the Lagrangian density is given by a sum of a kinetic term and a mass term. Free scalar field theory is the free theory describing a (for now, single) scalar field. It is the field-theoretic analogue of the mechanics of simple harmonic oscillators.

Quantization is easier for free theories because the interaction terms are absent. In particular, any discussion of renormalization group flow (which is crucial for regularizing the contributions from loop Feynman diagrams) can be completely avoided (hence we will see no BV quantizations appearing).

As emphasized in the proof of the existence of a P_0 factorization algebra structure on $\mathrm{Obs}^{\mathrm{cl}}$, the relevant algebraic characterization of a free field theory for us is that the associated L_∞ algebra is abelian (no brackets ℓ_n for $n \geq 2$). In particular, the dg structure on

$$\mathrm{Obs}^{\mathrm{cl}}(U) := \mathcal{O}(\mathcal{E}(U)) = \widehat{\mathrm{Sym}}(\mathcal{E}^\vee(U))$$

is simply extended (as a derivation) from the differential on $\mathcal{E}^\vee(U)$, with no higher Taylor coefficients (in the sense of Natalie's talk).

4.1. **The quantization procedure.** We will now proceed quantizing this classical algebra of observables via a factorizing envelope construction for a Heisenberg algebra. This can be seen as the mathematical incarnation of the usual canonical quantization procedure (valid for free theories) replacing the canonically conjugate (local) observables by the corresponding generators of the Heisenberg algebra.

Recall that we have a P_0-algebra

$$\mathrm{Obs}^{\mathrm{cl}}(U) = \left(\widehat{\mathrm{Sym}}(\mathcal{E}_c^!(U)), d\right)$$

(again, we are ignoring the functional analytic issue here). **For a free field**, the (+1)-Poisson bracket is determined on the level of generators (in degree 1) by the natural pairing

$$\{-,-\} : \mathcal{E}_c^!(U) \otimes \mathcal{E}_c^!(U) \to \mathbb{R}[1],$$

which is in turn induced from the (-1)-shifted pairing

$$\langle \cdot, \cdot \rangle : \mathcal{E}_c(U) \otimes \mathcal{E}_c(U) \to \mathrm{Dens}[-1] \xrightarrow{\int_M} \mathbb{R}[-1],$$

$$\langle \phi, \psi \rangle = \int_{x \in M} \langle \phi(x), \psi(x) \rangle_{\mathrm{loc}}.$$

We now construct the quantum theory be defining the Heisenberg algebra $\mathrm{Heis}(U)$ from the central extension of dg Lie algebras

$$0 \longrightarrow \mathbb{C}\cdot\hbar[-1] \longrightarrow \mathrm{Heis}(U) \longrightarrow \mathcal{E}_c^!(U)[-1] \longrightarrow 0,$$

with the Lie bracket

$$[\phi+\hbar a, \psi+\hbar b] = \hbar\langle\phi,\psi\rangle$$

of degree 0. Then we take the corresponding factorizing envelope as our definition of the quantum observables:

$$\mathrm{Obs}^q(U) := \left(\widehat{C}_*(\mathrm{Heis}(U)), d_{\mathrm{CE}}\right).$$

Proposition 4.1. *Associating $U \mapsto \mathrm{Obs}^q(U)$ is a BD factorization algebra quantizing $\mathrm{Obs}^{\mathrm{cl}}$.*

Proof. The quantum observables are given in terms of factorizing envelope, which is a completed Chevalley–Eilenberg chain complex:

$$\mathrm{Obs}^q(U) = \widehat{C}_*(\mathrm{Heis}(U)) = \widehat{\mathrm{Sym}}(\mathrm{Heis}(U)[1]).$$

As a graded vector space,

$$\widehat{\mathrm{Sym}}(\mathrm{Heis}(U)[1]) = \widehat{\mathrm{Sym}}(\mathbb{C}\hbar \oplus \mathcal{E}^\vee(U)) = \widehat{\mathrm{Sym}}(\mathcal{E}^\vee(U))[\![\hbar]\!] = \mathrm{Obs}^{\mathrm{cl}}(U)[\![\hbar]\!].$$

The quantum observables supported in U has a

- Product structure of degree 0 as a symmetric algebra,
- Poisson bracket of degree 1 extended as a derivation from the Lie bracket on $\mathrm{Heis}(U)$.

Since $\mathbb{C}\hbar[-1]$ is central, the Poisson bracket reduces modulo $\hbar$ to the Poisson bracket on classical observables. Moreover, the BD axiom

$$d(ab) = (da)b + (-1)^{|a|}a(db) + \hbar\{a,b\}$$

follows from definition (the Chevalley–Eilenberg differential is the sum of the internal differential and the Lie bracket).

Compatibility with the factorization algebra maps can be checked. □

References

[CG16a] K. Costello and O. Gwilliam. *Factorization Algebras in Quantum Field Theory Volume 1*. 2016. eprint: `https://people.math.umass.edu/~gwilliam/vol1may8.pdf`.

[CG16b] K. Costello and O. Gwilliam. *Factorization Algebras in Quantum Field Theory Volume 2*. 2016. eprint: `https://people.math.umass.edu/~gwilliam/vol2may8.pdf`.