FACTORIZATION ALGEBRAS, SYMMETRIES, AND QUANTIZATION

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Contents

Warning 0.1. I'm going to state a lot of theorems, definitions, etc imprecisely. I'll give citations for the precise statements if you want to look them up. Imprecise things will be put in quotes ("Theorem"), this just means I'm being lazy, not that the result isn't known.

Goals for this seminar:

(1) Understand the Costello-Gwilliam (CG) definition of a quantum field theory (QFT).
September-October
(2) Get a feeling for their quantum Noether theorem.
November-December

Goals for this *talk*:

(1) *Overview* the Costello-Gwilliam (CG) definition of a quantum field theory (QFT).
(2) Get a *vague* feeling for their quantum Noether theorem.

1. Field Theories after Costello-Gwilliam

There are many ways topologists describe a QFT.

- Segal: a *topological* QFT (TQFT) is a symmetric monoidal functor out of a bordism category
$$Z\colon \mathsf{Bord}_n^{\sqcup} \to \mathsf{Vect}^{\otimes}.$$
*this is like what we used last fall in Juvitop: Cobordism Hypothesis.
- Stolz-Teichner program: variation on Segal that works for non-topological, metrics involved but still bordism categories
- Vertex (operator) algebras: for 2d conformal field theories
- Conformal nets: also for 2d conformal field theories (Douglas/Henriques)
- Chiral algebras: algebraic geometric version of vertex algebras, but in all dimensions (Beilinson/Drinfel'd)
- Factorization algebra of observables, very similar to chiral algebras, but more from a topologists than a algebraic geometers perspective; works in all dimensions and don't need conformal (Costello-Gwilliam)

- $\mathcal{E}_n$-algebras: for TQFTs on $\mathbb{R}^n$
- Disk_n-algebras: slightly more general than $\mathcal{E}_n$-algebras, but still TQFT (Ayala/Francis)

Remark 1.1. These are not all known to be equivalent. (insert picture of what relationships are known)

In this seminar, we will be focused on the factorization algebra (CG) approach. Even within the CG approach, there are several reformulations[1] of the data of a field theory, all of which will be useful.

Within the CG framework, we will frequently want to switch between two realms

Realm A formal (pointed, elliptic) moduli problems
nice for geometric intuition

Realm B L_∞-algebras (i.e., homotopy Lie algebras)
algebraic and easier to work with for our computations

"Theorem" 1.2 (Lurie-Pridham)**.** *You can transport yourself between realms without losing any essence of being (i.e., there's an equivalence of categories).*

Future Talk. This will be covered in Ishan's talk in two weeks.

One direction is easier to say. Given a Lie algebra $\mathfrak{g}$, we get a stack $B\mathfrak{g}$ with underlying space a point and functions $\mathcal{O}_{B\mathfrak{g}} = C^\bullet_{\mathrm{Lie}}(\mathfrak{g})$.

Future Talk. Natalie's talk next week will cover Lie algebra (co)homology and such.

We usually want to consider QFTs over some space ($\sim$spacetime$\sim$) X. For today, you can think of X as a smooth manifold. We therefore have things like
- sheaves of formal moduli problems on X
- sheaves of L_∞-algebras on X

and such.

1.1. **Definition of a Field Theory.** The following is [1, Def. 4.2.0.4].

Definition 1.3 (Realm A: BV formalism)**.** A *classical field theory* on X is a formal, pointed, elliptic moduli problem $\mathcal{M}$ on X with a (-1)-shifted symplectic form ω.

Future Talk. Nat will talk about the other 5(?) formalisms on September 29th.

Think a classical field theory is a (shifted) symplectic "manfiold" living over X.

Example 1.4. The cotangent bundle $T^*X \to X$ is a symplectic manifold over X. The shifted cotangent bundle $T^*[-1]X$, with $\mathcal{O}_{T^*[-1]X} = \Gamma(X, \wedge^\bullet TX)$, is a classical field theory over X. Note the shift and it's lack of manifoldness.

Future Talk. Cameron will talk about this "cotangent theory," and some other examples, on October 6th.

"Definition" 1.5. The *classical observables* of a BV field theory $(\mathcal{M}, \omega)$ on X is the ring of functions $\mathcal{O}_\mathcal{M}$. We sometimes write $\mathrm{Obs}^{\mathrm{cl}}(X) = \mathcal{O}_\mathcal{M}$.

The CG prospective is to study a QFT by studying it's (quantum and classical) observables.

Future Talk. Wyatt will talk about what "observables" means in physics on September 29th.

Example 1.6. I said earlier that we'd go between realms. Say $\mathfrak{g}$ is a Lie algebra. Take the formal moduli problem $\mathcal{M} = B\mathfrak{g}$. Then, assuming we have an ω and things are elliptics, the classical observables are

$$\mathcal{O}_\mathcal{M} = \mathcal{O}_{B\mathfrak{g}} = C^\bullet_{\mathrm{Lie}}(\mathfrak{g}).$$

[1] These are all known to be equivalent.

1.2. **Quantization.** Note that functions $\mathcal{O}_{\mathcal{M}})$ is a commutative ring.

Problem

1) $\mathrm{Obs}^{\mathrm{cl}}$ only depended on $\mathcal{M}$, not on ω. If the observables should know about the whole field theory, it should know about both $\mathcal{M}$ and ω.

Solutions. Poisson algebras, quantization, factorization algebras, fancy things.

Definition 1.7. A *Poisson algebra* is a an associative algebra A equipped with a Lie bracket (called the Poisson bracket) that acts on A by derivations.

Proposition 1.8. *Let (M, ω_0) be a symplectic manifold. Then $\mathcal{O}_M$ is a Poisson algebra with Poisson bracket defined in terms of ω_0.*

*use ω_0 to define a vector field associated to each function (the Hamiltonian), then take the Lie bracket of vector fields.

Similarly, if $(\mathcal{M}, \omega)$ is a (-1)-shifted symplectic formal moduli problem, then $\mathcal{O}_{\mathcal{M}}$ is a (-1)-shifted Poisson algebra. From now on, when we refer to classical observables, it will be to the Poisson algebra of such. The benefit of Poisson algebras is that they are poised to be quantized, as in deformation theory.

Quantizing will give us $\mathrm{Obs}^q(X)$, "quantum (global) observables."

Future Talk. Jae's talk on October 27th will explain this quantization process for "free" theories.

1.3. **Factorization Algebras.** So far, we have kind of ignored our ∼spacetime∼ X. We can think about what we've been doing as working at a point $X = \mathrm{pt}$, or as working globally. But really, if everything is a sheaf on X, we'd like to have some LoCAL-tO-gLoBAl properties for our observables. This is part of where factorization algebras will come in.

Definition 1.9. A *factorization algebra* on X is a cosheaf on $\mathrm{Ran}(X)$, where $\mathrm{Ran}(X)$ is the Ran space

$$\mathrm{Ran}(X) = \operatorname*{colim}_I (U^I)$$

Future Talk. Zihong will give a more detailed/rigorous definition of a factorization algebra on October 13th.

Example 1.10. There's a special type of factorization algebra called *locally constant*. Locally constant factorization algebras on $\mathbb{R}^n$ are the same as $\mathcal{E}_n$-algebras (algebras over the little n-disks operad).

Future Talk. A TBD person will give us more examples of factorization algebras on October 20th.

Theorem 1.11 (Costello-Gwilliam)**.** *The classical and quantum observables define factorization algebras on X.*

2. Towards the Quantum Noether Theorem

Noether's theorem is supposed to be like

$$\text{symmetry of the theory} \implies \text{conserved quantity}$$

A little bit fancier/more precisely,

"Theorem" 2.1 (Noether)**.** *Given a differentiable symmetry generated by local actions, you get a corresponding conserved current.*

Let's break this down a bit in our new language. Given a field theory on X with classical observables $\mathrm{Obs}^{\mathrm{cl}}$, we make the following definition, which is [1, 12.5.0.1].

Definition 2.2. A *conserved current* in the field theory is a map of precosheaves

$$J\colon \Omega^{\bullet}_{X,c}[1] \to \mathrm{Obs}^{\mathrm{cl}}.$$

If you came in here today with some preconceived notion of what a conserved current is, a natural question to ask is

Question 1. How does this definition of a conserved current agree with my physics one?

That's a great question, which you should ask someone that isn't me. (Maybe Wyatt during the September 29 talk.)

So this is what we want to get. Input is a "symmetry generated by local actions." The symmetry part is going to come from a Lie algebra μ acting on our field theory.

Future Talk. Leon is going to tell us in detail about what it means for a Lie algebra to act on a BV theory on November 3rd.

For now, let's just play around. First, you might think

"μ should act on $\mathrm{Obs}^{\mathrm{cl}}$. That's the thing we care about."

We have $\mathrm{Obs}^{\mathrm{cl}} = \mathcal{O}_{\mathcal{M}}$, so it's a ring. We want the action to respect the multiplication. Assuming for a moment that $\mathcal{M}$ is manifoldy enough for things to work, we'd like a map

$$\mu \to \mathrm{Der}(\mathcal{O}_{\mathcal{M}}) = \mathrm{Vect}(\mathcal{M}).$$

But, we also want to take into consideration the symplectic structure ω on $\mathcal{M}$. So, we ask for a map

$$\rho\colon \mu \to \mathrm{Vect}^{\mathrm{symp}}(\mathcal{M}).$$

We will eventually want the action to be inner, which corresponds to acting by Hamiltonians. Recall that the symplectic structure ω defines a map

$$\mathrm{Ham}\colon \mathcal{O}_{\mathcal{M}}[-1] \to \mathrm{Vect}^{\mathrm{symp}}(\mathcal{M})$$

assigning to a function the corresponding Hamiltonian vector field. We would like a lift

$$\begin{tikzcd} & \mathcal{O}_{\mathcal{M}}[-1] \arrow[d, "\mathrm{Ham}"] \\ \mu \arrow[ur, dashed, "\tilde{\rho}"] \arrow[r, "\rho"'] & \mathrm{Vect}^{\mathrm{symp}}(\mathcal{M}) \end{tikzcd}$$

Claim. *There is an obstruction to creating such a lift, which is related to the obstruction to making the action of μ inner. The obstruction to making the action inner is an element $\alpha \in H^1_{\mathrm{Lie}}(\mu)$.*

Future Talk. The talk on November 10th will go through the details of these obstructions.

Now, the lifted map $\tilde{\rho}\colon \mu \to \mathcal{O}_{\mathcal{M}}[-1]$ is the same data as a map

$$\tilde{\rho}\colon \mu[1] \to \mathcal{O}_{\mathcal{M}}.$$

Since $\mathcal{O}_{\mathcal{M}}$ is a commutative ring, we get a map

$$\mathrm{Sym}(\mu[1]) \to \mathcal{O}_{\mathcal{M}}$$

of commutative algebras.

Again, we really want more than just commutative algebras. In particular, $\mathcal{O}_{\mathcal{M}}$ is a Poisson algebra.

Claim. *There exists a "twisted universal Poisson-enveloping algebra"* $U_\alpha^{P_0}(\mu)$.

We can now state a version of the classical Noether theorem, [1, Thm. 11.2.3.2].

Theorem 2.3 (Classical Noether Theorem - Poisson Version)**.** *Suppose we ahve a μ-action on a (-1)-shifted formal symplectic stack $\mathcal{M}$. Let $\alpha \in H^1_{\mathrm{Lie}}(\mu)$ be the obstruction to making the action inner. Then, there exists a canonical map*

$$U_\alpha^{P_0}(\mu) \to \mathcal{O}_{\mathcal{M}}$$

of Poisson algebras.

Remark 2.4. This was for $X = \mathrm{pt}$. If we want to work over X not a point, then we need to assume that the μ action is "local." (Remember the "local action" part of the Noether theorem.) We do this by requiring the action to go through "local functionals"

$$\mathcal{O}^{\mathrm{loc}}_{\mathcal{M}}[-1] \to \mathrm{Vect}(\mathcal{M})$$

instead of $\mathcal{O}_{\mathcal{M}}[-1]$.

Lemma 2.5. *If μ acts on a theory $(\mathcal{M}, \omega)$ on X, then $\mathcal{L} = \Omega^\bullet_X \otimes \mu$ does as well.*

This is because $\mathcal{L}$ is a resolution of the constant sheaf on X with stalk μ.

Working over $X \neq \mathrm{pt}$, we also need to involve factorization algebras. One can make a Poisson-factorization algebra $\mathbb{U}_\alpha^{P_0}(\mathcal{L})$, [1, Def. 12.4.1.1].

The following is [1, Thm. 12.4.1.2].

Theorem 2.6 (Classical Noether Theorem)**.** *Suppose that a local Lie algebra $\mathcal{L}$ acts on a classical field theory with Poisson-factorization algebra of observables* $\mathrm{Obs}^{\mathrm{cl}}$*, and that $\alpha \in H^1(C^\bullet_{\mathrm{red,loc,Lie}}(\mathcal{L}))$ is the obstruction to making this action inner.*

Then there is a homomoprhism of Poisson-factorization algebras

$$\mathbb{U}_\alpha^{P_0}(\mathcal{L}) \to \mathrm{Obs}^{\mathrm{cl}}.$$

Since both sides are now Poisson algebras, they can be quantized. This leads to the quantum Noether theorem.

Future Talk. Stating the quantum Noether theorem will be Sanath's talk on November 17. Sanath will give examples of the theorem on December 1st. Someone will outline the proof of the quantum Noether theorem on December 8th.

References

[1] Kevin Costello and Owen Gwilliam. *Factorization algebras in quantum field theory. Vol. 2*, volume 41 of *New Mathematical Monographs.* Cambridge University Press, Cambridge, 2021.