EXAMPLES OF FIELD THEORIES

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Introduction

In this talk we'll look at some examples of field theories, including the example of the free scalar, which we've seen a couple times before, as well as an interacting theory and a few examples of gauge theories. Our goal will be to get an idea of how one can do perturbation theory in Costello and Gwilliam's framework and to consider how some of these examples fit into a general construction called a cotangent theory.

1. Classical Field Theories and the BV Algorithm

references: [EW20]

From the beginning of this seminar, we've been considering two "realms" that describe quantum field theories within the Costello-Gwilliam framework:

- realm A: pointed elliptic formal moduli problems
- realm B: L_∞-algebras.

In Ishan's talk, we saw how to go between these realms using the Lurie-Pridham theorem, which gives an equivalence of categories. And in Natalia's talk, we defined what we mean by classical field theories and we learned about how to use the BV/BRST formalism to go from an action functional to the critical locus of our theory. We said that in the L_∞ algebra realm, we can define a classical field theory in the following way:

Definition 1.1. *A **classical field theory** is a local elliptic L_∞ algebra with a -3-shifted pairing.*

However, even if we stick to the L_∞ algebra realm, there are a couple of different formalisms we can consider.

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- **BRST formalism**: $(\mathcal{L}, S)$: in this formalism we have BRST fields given by sections of an L_∞ algebra, as well as a BRST action functional on the space of fields.
- **BV formalism**: in this formalism, we consider the derived critical locus $\mathrm{Crit}(S) = \mathcal{L} \oplus \mathcal{L}^![-3]$ as our BV fields, and additionally specify brackets for the L_∞ algebra.

Natalia showed us in her talk how we can go between these formalisms.

Recollection 1.2. *BV Algorithm:*

(1) take BRST fields $\mathcal{L}$ with $\mathcal{L}_0 = \mathrm{Lie}(\mathcal{G})$ and $\mathcal{L}_1 = T_{\phi_0}\mathcal{F}$
(2) add antifields to get $\mathcal{L} \oplus \mathcal{L}^![-3]$
(3) add differentials $l_n \colon \mathcal{L}^{\otimes n} \to \mathcal{L}^![-n-2]$ by the formula

$$S(\phi) = \sum_{k\leq 2} \frac{1}{k!} \int_M \langle l_{k-1}(\phi^{\otimes k-1}), \phi\rangle.$$

Note that usually the BRST fields are concentrated in degrees 0 and 1. We'll see how we should interpret these definitions more concretely when we discuss gauge theories. And recall that $\mathcal{L}!$ are the sections of $L^! = L^\vee \otimes \mathrm{Dens}_M$; see [CG20] 3.4.1.

2. Scalar Theories

In physics, scalar theories are often used to provide approximations or effective field theories to describe various phenomena, and one kind of particle they are supposed to describe well is the Higgs boson. Let M be an oriented manifold. Recall that for a scalar theory, our space of fields is taken to be $\mathcal{C}^\infty(M)$. Let's do two examples.

2.1. Free Scalar.
references: [CG20] §3.2.1, 4.5

Remark 2.1. *In classical physics, a free theory is a theory whose equations of motion are given by linear PDEs. The Lagrangian for such a theory is at most quadratic. For quantum theories, there is a similar description. For us, using the BV formalism, we can say our action is free if the local L_∞ algebra on our manifold is abelian and the action is quadratic* [EW20].

Let's revisit the example from Ishan's talk. We let $\phi \in \mathcal{F} = C^\infty(M)$. For now we will go ahead and identify $C^\infty(M)$ with its dual $\mathrm{Dens}(M)$, the space of densities, using the volume form on the manifold. The action functional for the free theory is

$$S(\phi) = \int_M \frac{1}{2}\phi \mathrm{D}\phi,$$

where D is the Laplacian.

Recollection 2.2. *In Wyatt's talk, we were introduced to the principle of stationary action. We saw that the Euler-Lagrange equations for an action S give the conditions on the fields ϕ that extremize $S(\phi)$. Wyatt derived the general formula for us, but for these specific examples you can derive the Euler-Lagrange equations by taking a small variation of your fields $\phi + \delta\phi$, computing $S(\phi + \delta\phi)$, and requiring that $\delta S = 0$.*

You can show that for the free scalar, the Euler-Lagrange equations are

$$\mathrm{D}\phi = 0$$

meaning that solutions are harmonic functions. This property of a free scalar should seem reasonable if you have taken enough physics classes to be convinced that many things in nature are well-modelled by coupled harmonic oscillators.

Remark 2.3. *If we also had a term in the action like $-\frac{1}{2}m^2\phi^2$, the theory would still be free, but then we'd recognize this term as a kinetic energy term and say that our particle has some nonzero mass m.*

One easy solution to these equations is zero. Let's classify the deformations of $\phi_0 = 0$ as a harmonic function on M. Remember from Ishan's talk that the complex to study in this situation is

$$C^\infty(M)[-1] \xrightarrow{\mathrm{D}} C^\infty(M)[-2]$$

We can see this from our algorithm by taking $\mathcal{L} = C^\infty(M)[-1]$. . Then adding the dual shifted by three, then identifying densities with smooth functions, we get $C^\infty(M)[-1] \oplus C^\infty(M)[-2]$. Finally, we need to specify the polydifferential operators. If we look at the action functional, we can see that in this free scalar case we have just have $l_1 = D$ and all $l_n = 0$ for $n \geq 0$.

Remark 2.4. *Really, the l_n differentials map land in* $\mathrm{Dens}(M)[-n-2]$ *since they are supposed to land in the dual, but we can just identify* $\mathrm{Dens}(M)$ *with $C^\infty(M)$ using the orientation on M, so we won't worry about this too much.*

2.2. **Interacting Scalar.** references: [CG20] §3.2.2, 4.5

Once we add in interactions, our equations of motion are no longer linear, and so they get hard to solve. This is some motivation for investigating the formal moduli problem of solutions around $\phi_0 = 0$ instead of trying to find the actual space of solutions.

The interacting model we'll discuss here is called ϕ^4 theory or a theory with a quartic interaction. One reason that this form of interaction is interesting is because in 4d space, you can show that the coupling constant (i.e. the coefficient of the interaction term) is dimensionless, which is important in the context of renormalization.

The action for this theory is

$$S(\phi) = \int_M \frac{1}{2}\phi \mathrm{D}\phi + \frac{1}{4!}\phi^4 \,\mathrm{dvol}_g.$$

Its Euler-Lagrange equations are thus

$$\mathrm{D}\phi + \frac{1}{3!}\phi^3 = 0.$$

The fields and antifields, as well as the differential $l_1 = D$ are the same as for the free theory, with

$$C^\infty(M)[-1] \xrightarrow{\mathrm{D}} C^\infty(M)[-2]$$

But now to fully describe the L_∞ algebra, we will do something more interesting than taking all higher brackets to be trivial. From our Taylor expansion formula, we see that we can account for the interaction term by defining the bracket l_3 to be

$$\begin{aligned} l_3 \colon C^\infty(M)^{\otimes 3} &\to C^\infty(M) \\ \phi \otimes \varphi \otimes \psi &\mapsto \phi\varphi\psi. \end{aligned}$$

2.2.1. *Maurer-Cartan Equations.* In Ishan's talk, we saw how to go between the two realms of formal moduli problems and L_∞ algebras. Roughly speaking, one direction of that process is the following:

$$\begin{aligned} &\mathsf{Lie} \simeq \mathsf{FMP} \\ &\mathcal{L} \mapsto ((R, \mathfrak{m}) \mapsto MC_\bullet(\mathcal{L} \otimes \mathfrak{m})). \end{aligned}$$

That is, $\mathcal{L}$ maps to a functor on Artinian algebras that takes R with maximal ideal $\mathfrak{m}$ to the Maurer-Cartan simplicial set. Recall the definition of the simplices.

Definition 2.5. *For $\mathcal{L}$ and $\mathfrak{m}$ as above, the Maurer-Cartan n-simplices are given by*

$$MC(\mathcal{L} \otimes \mathfrak{m})[n] = \{\alpha \in \mathcal{L} \otimes \mathfrak{m} \otimes \Omega^*(\Delta^n) \mid \sum_n \frac{1}{n!} l_n(\alpha^{\otimes n}) = 0\}$$

Claim 2.6. *For a non-dg Artinian algebra, the Maurer-Cartan equations impose the same conditions as the Euler-Lagrange equations.*

We saw at the end of Ishan's talk how the Maurer Cartan simplices could encode some higher order information in the form of homotopy coherences, if you feed in a dg Artinian algebra. In this example today, what we'll do instead is feed in a few choices of non-dg Artinian algebra to see how different algebras can help us resolve information when we enforce the Maurer-Cartan condition on the fields.

- The most basic thing we could do is try $R = k$ a field. Here, we get no new information, since the maximal ideal $\mathfrak{m} = 0$ and so $\mathcal{L} \otimes 0 \simeq 0$ and we have only zero fields left.
- The next thing we could do is try $R = k[\varepsilon]/(\varepsilon^2)$ the dual numbers, with their maximal ideal (ε). Morally, we should think of these as acting like tangent vectors. We can expand a field as $\phi = \phi_1\varepsilon$ with $\phi_1 \in C^\infty(M)$, and then the Maurer-Cartan equation tells us that
$$D(\phi_1\varepsilon) + \frac{1}{3!}(\phi_1\varepsilon)^3 = 0 \implies D\phi_1 = 0.$$
We've recovered the harmonic solutions, but nothing higher order.
- We'll get something similar if we try $R = k[\varepsilon]/(\varepsilon^3)$. But if we go one order further we'll finally get some interesting information; for $R = k[\varepsilon]/(\varepsilon^4)$, we can expand a field as $\phi = \varepsilon\phi_1 + \varepsilon^2\phi_2 + \varepsilon^3\phi_3$ and by matching orders of ε deduce that
$$\begin{aligned} D(\varepsilon\phi_1 + \varepsilon^2\phi_2 + \varepsilon^3\phi_3) + \frac{1}{3!}(\varepsilon\phi_1 + \varepsilon^2\phi_2 + \varepsilon^3\phi_3)^3 &= 0 \\ \implies D\phi_1 = 0,\ D\phi_2 = 0,\ D\phi_3 + \frac{1}{3}\phi_1^3 &= 0. \end{aligned}$$
The cool thing is that now we've reproduced an analog of the Euler-Lagrange equations in the form of a system of *linear* PDEs, which will be easier to solve!

What we were doing when we matched orders of ε above was picking out the different degree pieces from $\mathfrak{m}$. But the effect is that we have a perturbative expansion around $\phi_0 = 0$ for this theory.

3. Cotangent Theories

references: [CG20] §4.6.1, [EW20] 3.4

We'll discuss cotangent theories more or less to get terminology straight. We've been working with a definition of classical field theory in the L_∞ realm, but we also have a definition in the formal moduli problem realm:

Definition 3.1. *A **classical field theory** is a pointed, elliptic formal moduli problem with a -1-shifted pairing.* [CG20] *4.2.0.4*

Sometimes, we'll be handed a theory that already looks like this. Other times, we'll be handed a formal moduli problem that doesn't quite look like this, but we can take its cotangent theory to *make* it look like this.

So, let's define a cotangent theory. Let L be an elliptic L_∞ algebra on a manifold X and let $\mathcal{M}$ be its associated elliptic moduli problem from the Lurie-Pridham theorem. Let $L^! = L^\vee \otimes \mathrm{Dens}(X)$. Write $\mathcal{L}$ and $\mathcal{L}^!$, respectively, for their spaces of sections.

Definition 3.2. *Define $T^*[k]\mathcal{M}$ to be the elliptic FMP associated to the elliptic L_∞ algebra $\mathcal{L} \oplus \mathcal{L}^![k-2]$.*

Why is this something we would like to do? Mathematically, we like this because it automatically equips our L_∞ algebra with a pairing of degree $k-2$. Then to fit our definition of a classical field theory, we just take $k = -1$.

Definition 3.3. *Let $\mathcal{M}$ be an elliptic FMP corresponding to an elliptic L_∞ algebra $\mathcal{L}$. The **cotangent theory** associated to $\mathcal{M}$ is the elliptic FMP $T^*[-1]\mathcal{M}$, and its elliptic L_∞ algebra is $\mathcal{L}\oplus\mathcal{L}^![-3]$.*

This is really the same thing that Natalia told us about last week, except that we skip the last step of our algorithm and do not twist by the action. In her talk, we also saw a finite-dimensional example of such a theory which showed explicitly what the -1-shifted pairing looks like for the formal moduli problem—it comes from a -1-shifted symplectic structure.

Remark 3.4. *In Natalia's talk, we wrote $T^*[-3]M$ instead of $T^*[-1]M$, but we mean the same thing. The notation $T^*[-3]M$ agrees with* [EW20]*, but we'll use the latter notation to be consistent with Costello and Gwilliam.*

A bunch of important theories in physics arise from this cotangent theory construction, including the A and B-models of mirror symmetry and their half-twisted versions, as well as the $\beta\gamma$ model (a.k.a. "holomorphic Chern-Simons model"), and twisted supersymmetric gauge theories. We'll content ourselves with looking at the example self-dual Yang-Mills theory after we introduce gauge theories in general.

4. Gauge Theories

In his talk, Wyatt gave us some motivation for studying gauge theories. We saw, for example, that the theory of electrostatics was invariant under transformations of the electric potential $V \mapsto V+$ a constant, because all we can actually measure is a difference in potential. Meanwhile, if we work in a 4-dimensional spacetime, we can at least locally find an electromagnetic potential A which is such that $F = dA$, where F is the Faraday tensor, and we saw that F is invariant under a gauge transformation $A \mapsto A + d\lambda$.

4.1. **Setup.** Let's fix our setup for discussing gauge theories. Let G be a Lie group, M be an oriented manifold, and $P \to M$ be a principal G-bundle. We'll need some facts from differential geometry, which we can motivate but not give full arguments for here. For the precise arguments, we direct the reader to [KN].

Definition 4.1. *For G, P as above and a representation $\rho\colon G \to GL(V)$, the associated bundle over M is*

$$P \times_G V \coloneqq P \times V/(p, v) \sim (pg^{-1}, \rho(g)v)$$

Now we'll offer the following fact as an exercise.

Fact 4.2. *Sections of $P \times_G V$ are in bijection with G-equivariant maps $P \to V$.*

Definition 4.3. *The adjoint bundle of P is*

$$\mathfrak{g}_P \coloneqq P \times^{\mathrm{Ad}} \mathfrak{g}$$

where $\mathrm{Ad}\colon G \to GL(\mathfrak{g})$ is $g \mapsto (x \mapsto gxg^{-1})$ the adjoint action on the Lie algebra. This bundle is also often be written $\mathrm{ad}\, P$.

Definition 4.4. *Let $\mathcal{G} = \mathrm{Aut}(P)$ be the set of bundle automorphisms of P. This is the* ***gauge group****.*

Our goal now is to describe the L_∞ algebra corresponding to the formal moduli problem of connections on P near a fixed flat connection A_0. From Natalia's talk, we know that we should have

$$\mathcal{L}_1 = T_{A_0}\mathcal{F}, \qquad \mathcal{L}_0 = \mathrm{Lie}(\mathcal{G}).$$

Next we will identify these more concretely.

Remark 4.5. *See* [CG16] *§3.3.1 for a discussion of how to define the simplicial set corresponding to the formal moduli problem in this situation.*

Claim 4.6. $\mathcal{L}_1 = \Omega^1(M; \mathfrak{g}_P)[-1]$.

If one is familiar with the fact that the space of connections is affine, and that differences of connections can be represented by $\mathfrak{g}_P$-valued 1-forms, then this might seem reasonable already. The shift of $[-1]$ is there because we want to have the connections in degree zero since they are the data we are interested in for this theory. But we'll now give some more motivation for this claim.

Definition 4.7. *An (Ehresman)* ***connection*** *is a choice of horizontal subspace.*

What does this mean? Given our principal bundle with its projection map π, we can look at the derivative $d\pi$:

$$\begin{tikzcd} P \arrow[d, "\pi"] & \rightsquigarrow & TP \arrow[d, "d\pi"] \\ M & & TM. \end{tikzcd}$$

Definition 4.8. *Define $VP \coloneqq \ker d\pi \subset TP$. This is called the* ***vertical subspace****.*

This vertical subspace is canonically defined, but a set of complementary subspaces requires a choice. A connection A determines a horizontal subspace $HP \subset TP$, which is such that $HP_u \oplus VP_u = T_uP$ for each $u \in P$. And, specifying this connection is the same as specifying a map

$$A' \colon TP \to VP \text{ s.t. } A'|_{VP} = \mathrm{id}_{VP}.$$

Specifically, we can take $\ker A' = HP$.

With this definition in hand, we will cite another fact.

Fact 4.9. *Every $B \in \mathfrak{g}$ induces a fundamental vector field $B^* \in \mathscr{X}(TP)$, and the map $B \mapsto (B^*)_u$ defines an isomorphism $\mathfrak{g} \to VP_u$ for all $u \in P$.*

From here, we can start to believe the claim.

- A determines a 1-form $\omega_A \colon T_uP \to \mathfrak{g}$ with $\omega_A(X) = B$ where $(B^*)_u = d\pi(X)$.
- Given $X' \in \mathscr{X}(M)$, we can take the unique horizontal lift to $X \in \mathscr{X}(P)$ and apply ω_A to this.

In this way, A determines an element of $\Omega^1(M; \mathfrak{g}_P)$.

Claim 4.10. $\mathcal{L}_0 = \Omega^0(M; \mathfrak{g}_P)$.

First, we can offer some intuition for this claim. Note that the connections A and $A + d\lambda$, for $\lambda \in \Omega^0(M; \mathfrak{g}_P)$, have the same curvature. Hence it seems reasonable that our gauge transformations take this form.

Second, we can cite Fact 4.2 with $V = \mathfrak{g}$ to see that G-equivariant maps $P \to G$ define a section of $\mathfrak{g}_P$; i.e. an element of $\Omega^0(M; \mathfrak{g}_P)$.

4.2. **Chern-Simons Theory.** references: [CG16] §4.5, [CG20] §3.3.1, 3.3 [EW20] Ex. 3.3, 3.6, 3.8

Chern-Simons theory is an example of a 3D topological quantum field theory and has many uses in physics and mathematics, including in condensed matter physics and in knot theory.

We specialize now to M an oriented 3-manifold and G a simple Lie group. The Chern-Simons action functional is

$$S_{CS}(A) = \int_M \langle \frac{1}{2} A \wedge d_{A_0} A + \frac{1}{6}[A \wedge A] \rangle$$

where $\langle -, - \rangle$ is the Killing form on $\mathfrak{g}$ and where $[A \wedge A]$ means to wedge forms and bracket the Lie algebra parts; i.e. for $\alpha = \alpha_i \xi^i$ and $\beta = \beta_j \xi^j$, $[\alpha \wedge \beta] = \alpha_i \wedge \beta_j [\xi^i, \xi^j]$.

The corresponding Euler-Lagrange equations are $dA + [A \wedge A] = 0$, which holds if and only if the curvature $F_A = 0$. Hence this setup is appropriate to describe a formal moduli problem $\mathcal{M}$ parameterizing flat bundles on M near A_0.

By examining the action, we can see that $l_1 = d$ the de Rham differential and that $l_2 \neq 0$, but $l_n = 0$ for $n > 2$. We won't write out l_2 explicitly.

As part of our algorithm, we need to add on the antifields. To write these nicely, we can identify

$$(\wedge^k T^*M \otimes \mathfrak{g}_P)^! \cong \wedge^{3-k} T^*M \otimes \mathfrak{g}_P$$

using the orientation on M and the Killing form from $\mathfrak{g}$.

Putting all these pieces together and writing the differential l_1, we have

$$\Omega^0(M;\mathfrak{g}_P)[-1] \xrightarrow{d} \Omega^1(M;\mathfrak{g}_P)[-1] \xrightarrow{d} \Omega^2(M;\mathfrak{g}_P)[-1] \xrightarrow{d} \Omega^3(M;\mathfrak{g}_P)[-1]$$

That is, the BV complex in this case looks like the -1-shifted de Rham complex $\Omega^*(M;\mathfrak{g}_P)[-1]$. This comparison is a bit deceptive, since we also have the l_2 bracket—this is just harder to draw within the same diagram.

One more thing to observe in this example is that we have a concrete description of the -3-shifted pairing: define

$$\langle \alpha, \beta \rangle := \int_M \alpha \wedge \beta.$$

This definition involves integrating a top-degree form over our 3-manifold, but since the forms are shifted by 1, this amounts to a -3-shift overall. In general, we may not have such an intuitive way to write down the pairing.

4.3. **Yang-Mills Theory.** references: [EW20] Ex. 4.2

In physics, Yang-Mills theory with various choices of (in general, nonabelian) gauge group is used to study elementary particles and in particular was used to develop the Standard Model of particle physics. The geometry of the Yang-Mills equations is also very interesting mathematically.

We will now specialize to an oriented, semi-Riemannian 4-manifold (M, g). The BRST fields are analogous to those in the previous example. The Yang-Mills action functional is

$$S_{YM}(A) = \int_M \langle F_A \wedge \star F_A \rangle$$

where $F_A = dA + [A \wedge A]$ is the curvature and $\star$ is the Hodge star. If we expand the action as

$$S_{YM}(A) = \int_M \langle A \wedge (d \star dA + 2d \star [A \wedge A] + [A \wedge [A \wedge A]]) \rangle$$

we can see that $l_1 = d \star d$, $l_2(A, B) = [A \wedge B]$, $l_3 \neq 0$, and $l_n = 0$ for $n > 3$.

We can add antifields using a similar identification as in the previous example. The BV complex we arrive at looks like

$$\Omega^0(M;\mathfrak{g}_P)[-1] \xrightarrow{d} \Omega^1(M;\mathfrak{g}_P)[-1] \xrightarrow{d\star d} \Omega^3(M;\mathfrak{g}_P)[-1] \xrightarrow{d} \Omega^4(M;\mathfrak{g}_P)[-1].$$

Again we do not draw in the higher brackets l_2 and l_3. Note that in this example we do not get exactly a shifted de Rham complex—the 2-forms are missing. If we did this in general for an n-manifold, we would be "missing" even more intermediate groups.

Remark 4.11. *The Euler-Lagrange equations corresponding to this action are $d \star F_A = 0$. Set $G = U(1)$ and say M is a 4d spacetime manifold with a Lorentzian metric g. Combined with the Bianchi identity $dF_A = 0$ and the identifications of the components of the Faraday tensor with the components of the electric and magnetic fields that we saw in Wyatt's talk, we can reconstruct the four Maxwell's equations for electromagnetism in the vacuum.*

4.4. **Self-Dual Yang-Mills Theory.** references: [CG20] §3.3.2, 4.6.2, [EW20] Ex. 4.2

Let the setup be as in the previous example, but fixed in dimension four. We say that a connection A has self-dual curvature if its curvature $F_A = dA + [A \wedge A]$ is in the $+1$-eigenspace of the Hodge star operator $\star\colon \Omega^2(M) \to \Omega^2(M)$.

Write $\mathcal{M}(M, G)$ for the elliptic formal moduli problem parameterizing principal G-bundles on M with self-dual curvature. Then, self-dual Yang-Mills is the cotangent theory $T^*[-1]\mathcal{M}(M, G)$.

Let's go from this formal moduli problem to the derived critical locus. The L_∞ algebra controlling deformations of the pair (P, A_0) is

$$\mathcal{L} = \Omega^0(M;\mathfrak{g}_P) \xrightarrow{d_{A_0}} \Omega^1(M;\mathfrak{g}_P) \xrightarrow{d_-} \Omega^2(M;\mathfrak{g}_P)$$

where $d_- := \pi \circ d$ for $\pi = \mathrm{id} - \star$ the projection onto ASD connections. The BV complex for this example is thus $\mathcal{L} \oplus \mathcal{L}^![-3] =$

$$\begin{array}{ccccccc}
\Omega^0(M;\mathfrak{g}_P) & \xrightarrow{d_{A_0}} & \Omega^1(M;\mathfrak{g}_P) & \xrightarrow{d_-} & \Omega^2_-(M;\mathfrak{g}_P) & & \\
& & \oplus & & \oplus & & \\
& & \Omega^2_-(M;\mathfrak{g}_P) & \xrightarrow{d_{A_0}} & \Omega^3(M;\mathfrak{g}_P) & \xrightarrow{d_-} & \Omega^4(M;\mathfrak{g}_P)
\end{array}$$

where we again use orientation and the Killing form to rewrite the dual terms. We will not write down the action for this case, but note that there are no higher brackets.

References

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[EW20] Chris Elliott and Brian Williams. Masterclass on Supersymmetry and Twisting Lecture 1. October 2020.
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