Height of formal group laws as a complete invariant — Nir Gadish

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Recall: k field of char p, $f \in k[[x,y]]$

$$[p](t) = \underbrace{t +_f t +_f \cdots +_f t}_{p\text{-times}} = \cancel{pt} + \cdots$$

$$= \lambda t^{p^n} + \cdots$$

n = height of f.

More generally,

Def. $f \in R[[x,y]]$,

- $v_n \in R$ is the coeff. of t^{p^n} in the power series $[p]_f(t)$.
- f has height $\geq n$ if $v_0 = v_1 = \cdots = v_{n-1} = 0$.

- f has height exactly n if $v_n \in R^\times$.

- Sufficient to work with univ. FGL $f_{univ} \in L[\![x,y]\!]$, $v_n \in L$.

Ex. $p=2$,

$$f_{univ}(x,y) = x+y + t_1 xy + t_2(x^2y+y^2x) + t_3(2x^3y+3x^2y^2+2xy^3) + \ldots$$

$$[2](x) = f_{univ}(x,x)$$
$$= \underbrace{2x}_{v_0} + \underbrace{t_1}_{v_1} x^2 + 2t_2 x^3 + \underbrace{7t_3}_{} x^4 + \ldots$$

$$2^0 \qquad 2^1 \qquad v_2 \ \dots \quad 2^2$$

Prop. Up to decomposables

$$v_n \equiv (p^{p^n-1}-1)\, t_{p^n-1}$$

Pf. Working mod $t_1, \dots, t_{k-1}$ $(k = p^n - 1)$

$$f_{univ}(x,y) \equiv x+y + t_k \sum \underbrace{\frac{\binom{p^n}{i}}{p} x^i y^{p^n-i}}_{\text{forced by assoc.}} + \dots$$

$$= x+y + \frac{t_k}{p}\left((x+y)^{p^n} - x^{p^n} - y^{p^n}\right)$$

By induction $\forall m$,

$$[m](x) = mx + \frac{t_k}{p}\left((mx)^{p^n} - mx^{p^n}\right) + \dots$$

when $p = m$,

$$[p](x) = px + \underbrace{\frac{t_k}{p}(p^{p^n} - p)}\, x^{p^n} + \dots$$

$$v_n \equiv (p^{p^n - 1} - 1)\, t_k$$

up to decomposable

<u>p-locally</u>, ($l \neq p$ is invertible)

$$v_n \sim t_{p^n - 1} \quad \text{up to invertible elements}$$

$\rightsquigarrow$ get new coords. for p-localized Lazard ring

$$L_{(p)} \cong \mathbb{Z}_{(p)}[t_1', t_2', \ldots]$$

with $\boxed{t'_{p^n-1} = v_n}$

(we make v_n to be part of the free parameters defining a FGL)

<u>Cor.</u> If k is a field of char p, $\exists$ FGL of every height $1 \leq n \leq \infty$.

<u>Pf.</u> $n = \infty \rightsquigarrow f(x,y) = x+y$.

$n < \infty \rightsquigarrow$ define f by sending

$t_1, \dots, t_{p^n-2} \mapsto 0$

$v_n = t_{p^n-1} \mapsto 1$

$\Longrightarrow$ has height exactly n.

Note: height $\geq n$ is an iso invariant, moreover this a Zariski-local property.

$\rightsquigarrow$ Can define height $\geq n$ for all formal groups,
"if in all coordinate charts has height $\geq n$."

$\rightsquigarrow$ a stratification of moduli stack $\mathcal{M}_{FG} \times \operatorname{Spec} \mathbb{Z}_{(p)}$

$$\mathcal{M}_{FG} \times \operatorname{Spec} \mathbb{Z}_{(p)} \supseteq \cdots \subseteq \mathcal{M}_{FG}^{\geq n+1} \subseteq \mathcal{M}_{FG}^{\geq n}$$

<u>Ex</u>

- height $0 \rightsquigarrow v_0$ is invertible, $v_0 = p$

  $\Longrightarrow$ char 0, only one fgl $F(x,y) = x+y$.

  $\Longrightarrow \mathcal{M}_{FG}^{0} = B\operatorname{Aut}(x+y)_{\text{over } \mathbb{Q}}$

- height $\infty \rightsquigarrow v_0 = 0 \Longrightarrow p = 0$

  char p !

and we see below - iso to $f(x,y) = x+y$.

$\Longrightarrow \mathcal{M}^{\infty}_{FG} = B\mathrm{Aut}(x+y)$ over $\mathbb{F}_p$

- What happens at $0 < n < \infty$?

Thm (Lazard) Over $k = \bar{k}$ char p

$f \cong f' \iff$ same height n.

really, an iso $f \longrightarrow f'$

is defined by solving

separable poly equations.

- Standard fgl of height n:

$F(x,y) = x+y + \sum \frac{\binom{p}{i}}{p} x^i y^{p-i}$

- Sufficient to find iso $f \xrightarrow{g} F$

$$g(t) = b_0 t + b_1 t^2 + b_2 t^3 + \dots$$

$k^\times$

- Construct g by a sequence of approx.

$$f = f_1 \xrightarrow{g_1} f_2 \xrightarrow{g_2} f_3 \to \dots$$

where $f_m \equiv F \mod (x,y)^m$

so that $g_\infty = \dots \circ g_3 \circ g_2 \circ g_1$

would converge and give

un iso $f' \longrightarrow F$.

<u>Lemma:</u> If f and f' are two FGL
and $f \equiv f' \mod (x,y)^m$ –
then next term $f = f' + \lambda \sum \frac{\binom{m}{i}}{d_m} x^i y^{m-i} \mod (x,y)^{m+1}$

where $d_m = \begin{cases} p & m = p^k \\ 1 & \text{otherwise.} \end{cases}$

[Again, Consequence of assoc.]

So we have – $f_m(x,y) \equiv F + \lambda \sum \frac{\binom{m}{i}}{d_m} x^i y^{m-i}$

And we want – $\mod (x,y)^{m+1}$

$g_m(t)$ a change of coords

s.t. $f_{m+1} = g_m^{-1} f_m(g_m(x), g_m(y))$

turns $\lambda \rightsquigarrow 0$.

<u>Case 1</u> : m not p^k.

$$f_m \equiv F + \lambda\left((x+y)^m - x^m - y^m\right) + \ldots$$

$$g_m(t) = t + ct^m$$

(check) $\rightsquigarrow$ $$f_{m+1} \equiv F + (\lambda - c)\left((x+y)^m - x^m - y^m\right) + \ldots$$

Take $\boxed{c = \lambda}$. – this can be done over any ring! (so can always ignore) such terms!)

Case 2: $m = p^k$, $k < n$

lower than height

$$f_m(x,y) \equiv x + y + \underbrace{\lambda}_{= v_k = 0} \sum \frac{\binom{m}{i}}{p} x^i y^{m-i} + \dots$$

Already good! (by height $> k$) ✓

Case 3: $m = p^n$,

$$f_m = x + y + \underbrace{\lambda}_{= v_n} \sum \frac{\binom{m}{i}}{p} x^i y^{m-i} + \dots$$

invertible.

Want $\lambda \rightsquigarrow 1$.

Take $g_m(t) = c_0 t$,

invertible

invertible

$$\rightsquigarrow c_0^{-1} f_m(c_0 x, c_0 y) = x + y + c_0^{-1} \lambda \sum \frac{\binom{m}{i}}{p} c_0^m x^i y^{m-i}$$

$$= x + y + \underbrace{c_0^{m-1} \lambda}_{\text{want } = 1} \sum \ldots$$

take $\boxed{c_0^{p^n - 1} = \frac{1}{\lambda}}$ – solve this separable equation. ✓

Case 4: $m = p^{n+k}$ (hardest)

Key idea – use the arithmetic of f_m

[it has commutativity]

(assoc., inverses, ...)

$$g_m(t) = f_m(t, ct^{p^k})$$

$$= t + ct^{p^k} + \text{HOT}$$

get a correction term

$$\rightsquigarrow \quad f_{m+1} = f_m + (c^{p^k} - c) \sum \frac{\binom{p^n}{i}}{p} x^{ip^k} y^{m-ip^k} + \dots$$

comes from term $- f_m \ni \left(\sum \frac{\binom{p^n}{i}}{p} x^i y^{p^n - i} \right)^{p^k}$

Want to eliminate

$$f_m = F + \lambda \sum \frac{\binom{p^{n+k}}{j}}{p} x^j y^{m-j} + \dots$$

Take - $\boxed{c^{p^k} - c = -\lambda}$ a sep.

equation.

+ Need the Combinatorial lemma:

$$\binom{p^{n+k}}{j} \equiv \begin{cases} \binom{p^n}{i} & j = ip^k \\ 0 & \text{otherwise} \end{cases} \mod p^n.$$

Idea: Look at $C_{p^n} = G$

it acts on $\binom{G}{i}$ subsets, $\left|\binom{G}{i}\right| = \binom{p^n}{i}$

and count fixed points/orbits of the

subgroup $G' = p\mathbb{Z}/p^n\mathbb{Z}$.