Goerss-Hopkins obstruction theory

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The basic cast of characters

This is a talk on chromatic homotopy theory, so:

- Fix a prime p and a height h.
- k is a perfect field of characteristic p.
- Γ is a formal group law of height h over k.

Automorphisms of formal group laws

Last time, Lucy told us about the Morava stabilizer group

$$\mathbb{G} = \mathsf{Aut}(k, \Gamma) = \mathsf{Aut}(\Gamma) \rtimes \mathsf{Gal}$$

of automorphisms of the formal group law Γ.

Lubin-Tate theories

We also have Lubin-Tate theories.

We briefly recall their construction:

- Lubin-Tate: the formal group law Γ admits a universal deformation to a formal group law over $W(k)[\![u_1, \dots, u_{h-1}]\!]$.
- This is Landweber flat.
- Hence, the Landweber exact functor theorem gives a even-periodic homology theory or spectrum $E(k, \Gamma)$.

Action of the Morava stabilizer group

By construction, the Morava stabilizer group $\mathbb{G}$ acts on

$$\pi_* E(k, \Gamma) = W(k)[\![u_1, \ldots, u_{h-1}]\!][u^{\pm 1}].$$

Question

Does this lift to an action of $\mathbb{G}$ on the spectrum $E(k, \Gamma)$?

Homotopy fixed points of Morava E-theory

Why might we want an action of $\mathbb{G}$ on $E := E(k, \Gamma)$?

Theorem (Devinatz-Hopkins)

$E^{h\mathbb{G}} \simeq S^0_{K(h)}$.

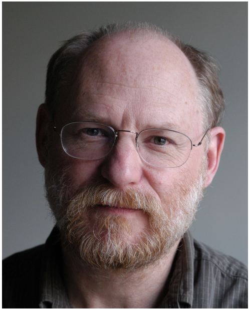
Homotopy fixed points of Morava E-theory

- Let $F < \mathrm{Aut}(\Gamma)$ be a maximal finite subgroup. Then $EO_h := E^{hF}$ is a *higher real K-theory*.
- You can also get localizations of TMF.
- Also, we expect that $S^0_{K(h)} \simeq E^{h\mathbb{G}}$ can be approximated in terms of E^{hF} for various finite subgroups $F < \mathbb{G}$.
  - When $h = 1, 2$, there exist resolutions of $S^0_{K(h)}$ by certain E^{hF}.
  - Moreover, the homotopy groups of E^{hF} are amenable to calculation for example via homotopy fixed point spectral sequences.

Lifting algebra maps to spectra

- We want to lift an action on homotopy groups to an action on spectra.
- This problem is hard!

Solution?

"Make the problem harder!"

Making the problem harder

We refine the problem of lifting the action of $\mathbb{G}$ to the spectrum E in two ways:

1. Demand that $\mathbb{G}$ acts on E via $\mathbb{E}_\infty$-ring automorphisms.
2. Study the entire moduli space of $\mathbb{E}_\infty$-ring realizations of E, i.e.,

$$\mathcal{M}_{\mathbb{E}_\infty}(E) = \left\{ \begin{array}{c} \mathbb{E}_\infty\text{-rings } A \text{ such that } A \simeq E \\ \text{as homotopy associative rings} \end{array} \right\}.$$

Remark

Implicit in (1) is the claim that E is an $\mathbb{E}_\infty$-ring. A priori, we only know that E is a homotopy commutative ring from the Landweber construction. Most of the work will be showing that it is in fact $\mathbb{E}_\infty$.

The Goerss-Hopkins-Miller theorem

$$\mathcal{M}_{\mathbb{E}_\infty}(E) = \{\mathbb{E}_\infty\text{-rings } A \simeq E\}$$

Our main theorem today:

Theorem (Goerss-Hopkins-Miller)

$$\mathcal{M}_{\mathbb{E}_\infty}(E) \simeq B\mathbb{G}.$$

- In other words, there exists a unique $\mathbb{E}_\infty$-ring structure on E, and $\mathrm{Aut}_{\mathbb{E}_\infty}(E) \cong \mathbb{G}$.
- Our tool to prove this is Goerss-Hopkins obstruction theory.

Other approaches, or a history of Juvitop

There have been other approaches to endow Morava E-theory with additional structure.

- A precursor is the Hopkins-Miller theorem, which studied $\mathbb{A}_\infty$-ring structures on E.
  - Danny gave a Juvitop talk on this in Fall 2016!
- There are other $\mathbb{E}_\infty$-obstruction theories developed to study Morava E-theory, e.g., Robinson's Γ-homology.
  - Juvitop talk, Spring 2017!
- More recently, Lurie gave a different construction of Lubin-Tate theory as the solution to a moduli problem in spectral algebraic geometry. The $\mathbb{E}_\infty$-structure in this case is automatic.
  - Hood gave a Juvitop talk on this in Spring 2018!

Outline for the technical part of this talk

Primer on synthetic spectra

Goerss-Hopkins obstruction theory

Application to Morava E-theory

Why synthetic spectra?

We'll follow a modern presentation of Goerss-Hopkins obstruction theory set out by Hopkins-Lurie and Pstrągowski-VanKoughnett, which situates moduli problems for commutative ring spectra in the context of synthetic spectra.

- Synthetic spectra is a technique for working with *resolutions*.

Slogan

Goerss-Hopkins theory is ($\mathbb{E}_\infty$-)Postnikov tower theory internal to synthetic spectra.

Definition of synthetic spectra

- Let E be an Adams-type homology theory (think Morava E-theory).
- Let $\mathcal{S}p_E^{\mathrm{fp}}$ be the category of finite spectra P such that E_*P is a finitely generated projective E_*-module.
- Equip $\mathcal{S}p_E^{\mathrm{fp}}$ with a Grothendieck topology where $\{P \to Q\}$ is a covering iff $E_*P \to E_*Q$ is surjective.

Definition

A (hypercomplete connective) synthetic spectrum is a product-preserving hypercomplete sheaf of spaces on $\mathcal{S}p_E^{\mathrm{fp}}$.

- Denote the category of synthetic spectra by $\mathcal{S}yn_E$.

Basic features of $\mathcal{S}yn_E$

- $(\mathcal{S}yn_E, \otimes, \mathbb{1})$ is a symmetric monoidal category, where
  - $\otimes$ is given by Day convolution, and
  - the monoidal unit $\mathbb{1}$ is the *synthetic analogue* $\nu S_E^0 := L(\mathrm{Map}_{\mathcal{S}p}(-, S_E^0))$ of the E-local sphere.
- There is an autoequivalence of $\mathcal{S}yn_E$, compatible with the symmetric monoidal structure, defined by sending X to $X[1] = X \circ \Sigma^{-1}$.
  This gives rise to a second grading on $\mathcal{S}yn_E$.

The comparison map τ

- There is a map
$$\tau : \Sigma X[-1] \to X$$
for any $X \in \mathcal{S}yn_E$ induced by the adjoint of the comparison map $X(\Sigma P) \to \Omega X(P)$ for $P \in \mathcal{S}p_E^{\mathrm{fp}}$.
- We think of τ as a parameter controlling the behaviour of the category $\mathcal{S}yn_E$. Heuristically:

$$\mathcal{S}yn_E^{\mathrm{per}} \xleftarrow{\text{invert } \tau} \mathcal{S}yn_E \xrightarrow{\text{kill } \tau} \mathcal{S}yn_E^{\heartsuit}$$

More properties of $\mathcal{S}yn_E$

$$\mathcal{S}yn_E^{\mathrm{per}} \xleftarrow{\text{invert } \tau} \mathcal{S}yn_E \xrightarrow{\text{kill } \tau} \mathcal{S}yn_E^{\heartsuit}$$

- $\mathcal{S}yn_E^{\mathrm{per}} \simeq \mathcal{S}p_E$ as symmetric monoidal categories.
- $\mathcal{S}yn_E^{\heartsuit} \simeq \mathcal{C}omod_{E_*E}$ as symmetric monoidal categories.

We'll also want:

- $\mathcal{S}yn_E$ is *complete*: every Postnikov tower converges.

Remark

This last property is not automatic. It holds when E is Lubin-Tate theory using somewhat deep results related to vanishing lines in the E-based Adams SS.

Primer on synthetic spectra

Goerss-Hopkins obstruction theory

Application to Morava E-theory

Potential n-stages

Definition

A potential n-stage is an $\mathbb{E}_\infty$-algebra R in $\mathcal{M}od_{\mathbb{1}_{\leq n}}(\mathcal{S}yn_E)$ such that $\mathbb{1}_{\leq 0} \otimes_{\mathbb{1}_{\leq n}} R$ is discrete.

- Denote the category of potential n-stages by $\mathcal{M}_n$.
- Extension of scalars along $\mathbb{1}_{\leq n} \to \mathbb{1}_{\leq n-1}$ induce maps $\mathcal{M}_n \to \mathcal{M}_{n-1}$.
- Hence we get a tower

$$\mathcal{M}_\infty \to \cdots \to \mathcal{M}_n \to \mathcal{M}_{n-1} \to \cdots \to \mathcal{M}_0.$$

- Under our completeness assumption, $\mathcal{M}_\infty \simeq \lim \mathcal{M}_n$.

Potential n-stages

Examples

- A potential 0-stage is just an ordinary commutative algebra in E_*E-comodules.
- A potential ∞-stage is a E-local $\mathbb{E}_\infty$-ring spectrum.
- Moreover, the map $\mathcal{M}_\infty \to \mathcal{M}_0$ sends R to E_*R.

- So these potential stages interpolate between algebra and geometry.

Postnikov tower theory for commutative algebras

We want to relate $\mathcal{M}od_{\mathbb{1}_{\leq n}}$ with $\mathcal{M}od_{\mathbb{1}_{\leq n-1}}$.

- Note $\mathbb{1}_{\leq n}$ is a square-zero extension of $\mathbb{1}_{\leq n-1}$, so that there is a pullback square in $\mathcal{C}Alg$ of the form

$$\begin{array}{ccc} \mathbb{1}_{\leq n} & \longrightarrow & \mathbb{1}_{\leq n-1} \\ \downarrow & & \downarrow i \\ \mathbb{1}_{\leq n-1} & \xrightarrow[i_0]{} & \mathbb{1}_{\leq n-1} \oplus \Sigma^{n+1}\pi_0\mathbb{1}[-n]. \end{array}$$

Postnikov tower theory for module categories

Proposition

There is a pullback square of categories

$$\begin{array}{ccc} \mathcal{M}od_{\mathbb{1}_{\leq n}} & \longrightarrow & \mathcal{M}od_{\mathbb{1}_{\leq n-1}} \\ \downarrow & \lrcorner & \downarrow{\scriptstyle i^*} \\ \mathcal{M}od_{\mathbb{1}_{\leq n-1}} & \xrightarrow[i_0^*]{} & \mathcal{M}od_{\mathbb{1}_{\leq n-1} \oplus \Sigma^{n+1}\pi_0 \mathbb{1}[-n]}. \end{array}$$

Modules with sections

- Define a functor $\Theta : \mathcal{M}od_{\mathbb{1}_{\leq n-1}} \to \mathcal{M}od_{\mathbb{1}_{\leq n-1}}$ by
$$\Theta(M) = i_* i_0^* M.$$
- The underlying spectrum of $\Theta(M)$ is $M \oplus \Sigma^{n+1}(M \otimes_{\mathbb{1}_{\leq n-1}} \pi_0 \mathbb{1}[-n])$.
- Let $p : \mathbb{1}_{\leq n-1} \oplus \Sigma^{n+1}\pi_0 \mathbb{1}[-n] \to \mathbb{1}_{\leq n-1}$ be the projection. The unit of the adjunction $p^* \dashv p_*$ induces a map
$$\pi : \Theta M = i_* i_0^* M \to i_* p_* p^* i_0^* M \simeq \mathrm{id}_* \, \mathrm{id}^* \, M \simeq M.$$

Definition

$\Theta\mathcal{S}ect_{\mathbb{1}_{\leq n-1}}$ is the category of $\mathbb{1}_{\leq n-1}$-modules equipped with a section $s : M \to \Theta M$ of π.

Modules with sections

Theorem

$\mathcal{M}od_{\mathbb{1}_{\leq n}} \simeq \Theta \mathcal{S}ect_{\mathbb{1}_{\leq n-1}}$ *as symmetric monoidal categories.*

Proof.

- By the previous proposition,

$$\mathcal{M}od_{\mathbb{1}_{\leq n}} \simeq \{(M, N \in \mathcal{M}od_{\mathbb{1}_{\leq n-1}}, \alpha : i^* N \xrightarrow{\sim} i_0^* M)\}.$$

- Let $\hat{\alpha} : N \to \Theta M$ be adjoint to α.
- Fact: $\alpha : i^* N \to i_0^* M$ is an equivalence iff $\pi \circ \hat{\alpha}$ is an equivalence (because $\pi \circ \hat{\alpha} = p^* \alpha$ and p^* is conservative).

Modules with sections

Theorem

$\mathcal{M}od_{\mathbb{1}_{\leq n}} \simeq \Theta\mathcal{S}ect_{\mathbb{1}_{\leq n-1}}$ as symmetric monoidal categories.

Proof.

- Therefore, we have

$$\begin{aligned} \mathcal{M}od_{\mathbb{1}_{\leq n}} &\simeq \{(M, N, \alpha : i^* N \xrightarrow{\sim} i_0^* M)\} \\ &\simeq \{(M, N, \hat{\alpha} : N \to \Theta M) \mid \pi \circ \hat{\alpha} \text{ is } \simeq\} \\ &\to \Theta\mathcal{S}ect_{\mathbb{1}_{\leq n-1}} \end{aligned}$$

  sending $(M, N, \hat{\alpha})$ to $(M, \hat{\alpha} \circ (\pi \circ \hat{\alpha})^{-1}) \in \Theta\mathcal{S}ect_{\mathbb{1}_{\leq n-1}}$.
- The fiber of this functor is identified with $\{N \mid N \simeq M\}$, which is contractible.

□

Case when R is a potential $(n-1)$-stage

Lemma

Let R be a potential $(n-1)$-stage. Then the map $\pi : \Theta R \to R$ is a square-zero extension of R by $\Sigma^{n+1}\pi_0 R[-n]$.

Proof.

- The fiber of the map

$$\pi : \Theta R \simeq R \oplus \Sigma^{n+1}(R \otimes_{\mathbb{1}_{\leq n-1}} \pi_0 \mathbb{1}[-n]) \to R$$

  is just $\Sigma^{n+1}(R \otimes_{\mathbb{1}_{\leq n-1}} \pi_0 \mathbb{1}[-n])$.
- By definition of a potential $(n-1)$-stage, this is concentrated in a single degree.

□

- Consequently, a lift of R to a potential n-stage exists iff this extension admits a section by the previous theorem.

Cotangent complexes

- Square-zero extensions are controlled by cotangent complexes.
- More precisely:

$$\{\text{square-zero extensions of } R \in \mathcal{C}Alg_A \text{ by } M \in \mathcal{M}od_R\}$$
$$\simeq \pi_0 \operatorname{Map}_{\mathcal{M}od_R}(\mathbb{L}_{R/A}, \Sigma M).$$

- An extension is split iff the classifying map from the cotangent complex is null.

Obstructions to lifting objects

Theorem (Goerss-Hopkins)

Let R be a potential $(n-1)$-stage. There is an obstruction in the André-Quillen cohomology group

$$\mathrm{Ext}^{n+2,n}_{\mathcal{M}od_{\pi_0 R}(\mathcal{S}yn_E^{\heartsuit})}(\mathbb{L}_{\pi_0 R/\pi_0 \mathbb{1}}, \pi_0 R)$$

which vanishes iff R can be lifted to a potential n-stage.

Proof.

- R lifts to a potential n-stage iff $\pi : \Theta R \to R$ admits a section iff the map

$$\mathbb{L}_{R/\mathbb{1}_{\leq n-1}} \to \Sigma^{n+2}\pi_0 R[-n]$$

classifying the square-zero extension π is null.

Obstructions to lifting objects

Proof.

- Next, we describe the group of maps $\mathbb{L}_{R/\mathbb{1}_{\leq n-1}} \to \Sigma^{n+2}\pi_0 R[-n]$ in terms of algebra.

$$\begin{aligned}
&\pi_0 \operatorname{Map}_{\mathcal{M}od_R(\mathcal{S}yn_E)}(\mathbb{L}_{R/\mathbb{1}_{\leq n-1}}, \Sigma^{n+2}\pi_0 R[-n]) \\
&\simeq \pi_0 \operatorname{Map}_{\mathcal{M}od_{\pi_0 R}(\mathcal{S}yn_E^{\heartsuit})}(\pi_0 R \otimes_R \mathbb{L}_{R/\mathbb{1}_{\leq n-1}}, \Sigma^{n+2}\pi_0 R[-n]) \\
&\simeq \pi_0 \operatorname{Map}_{\mathcal{M}od_{\pi_0 R}(\mathcal{S}yn_E^{\heartsuit})}(\pi_0 \mathbb{1} \otimes_{\mathbb{1}_{\leq n-1}} \mathbb{L}_{R/\mathbb{1}_{\leq n-1}}, \Sigma^{n+2}\pi_0 R[-n]) \\
&\simeq \pi_0 \operatorname{Map}_{\mathcal{M}od_{\pi_0 R}(\mathcal{S}yn_E^{\heartsuit})}(\mathbb{L}_{\pi_0 R/\pi_0 \mathbb{1}}, \Sigma^{n+2}\pi_0 R[-n]) \\
&=: \operatorname{Ext}^{n+2,n}_{\mathcal{M}od_{\pi_0 R}(\mathcal{S}yn_E^{\heartsuit})}(\mathbb{L}_{\pi_0 R/\pi_0 \mathbb{1}}, \pi_0 R).
\end{aligned}$$

□

Obstructions to lifting objects

Corollary

Let $A \in \mathcal{C}Alg(\mathcal{C}omod_{E_*E})$. *There exists an inductively defined sequence of obstructions valued in André-Quillen cohomology groups* $\mathrm{Ext}^{n+2,n}_{\mathcal{M}od_A(\mathcal{C}omod_{E_*E})}(\mathbb{L}_{A/E_*}, A)$ *for* $n \geq 1$*, which vanish iff there is an* $\mathbb{E}_\infty$*-ring spectrum* R *such that* $E_*R \cong A$ *as comodule algebras.*

Obstructions to lifting maps

There's a version of this machinery for lifting maps too, which is useful in determining the uniqueness of lifting objects.

Here is the setup:

- R, S potential n-stages.
- $\phi : uR \to uS$ maps of corresponding potential $(n-1)$-stages.

Question

Does ϕ lift to a map $R \to S$?

A similar argument as before shows. . .

Obstructions to lifting maps

Theorem (Goerss-Hopkins)

Let R and S be potential n-stages and $\phi : uR \to uS$ a map of corresponding potential $(n-1)$-stages. Then,

(a) *There is an obstruction in $\mathrm{Ext}^{n+1,n}_{\mathcal{M}od_{\pi_0 R}(\mathcal{S}yn_E^{\heartsuit})}(\mathbb{L}_{\pi_0 R/\pi_0 \mathbb{1}}, \pi_0 S)$ which vanishes iff ϕ lifts to a map $R \to S$.*

(b) *In this case, the space of lifts of ϕ is*

$$\mathrm{Map}_{\mathcal{M}od_{\pi_0 R}(\mathcal{S}yn_E^{\heartsuit})}(\mathbb{L}_{\pi_0 R/\pi_0 \mathbb{1}}, \Sigma^n \pi_0 S[-n]).$$

Corollary

Let R and S be E-local $\mathbb{E}_\infty$-rings, and let $A = E_* R$ and $B = E_* S$. Given a map $\phi : A \to B$ of commutative algebras in $E_* E$-comodules, there exists an inductively defined sequence of obstructions valued in $\mathrm{Ext}^{n+1,n}_{\mathcal{M}od_A(\mathcal{C}omod_{E_* E})}(\mathbb{L}_{A/E_*}, B)$ which vanishes iff there is an $\mathbb{E}_\infty$-ring map $\tilde{\phi} : R \to S$ such that $E_* \tilde{\phi} = \phi$.

The mapping space spectral sequence

Corollary (Goerss-Hopkins)

Suppose we're given a morphism $\phi : R \to S$ of $\mathcal{S}yn_E^{\mathrm{per}}$. There is a first quadrant spectral sequence converging conditionally to $\pi_{t-s}(\mathrm{Map}_{\mathcal{C}Alg(\mathcal{S}yn_E)}(R, S))$ with E_1-page given by

$$E_1^{0,0} = \mathrm{Map}_{\mathcal{C}Alg(\mathcal{S}yn_E^{\heartsuit})}(\pi_0 R, \pi_0 S),$$
$$E_1^{s,t} = \mathrm{Ext}^{2s-t,s}_{\mathcal{M}od_{\pi_0 R}(\mathcal{S}yn_E^{\heartsuit})}(\mathbb{L}_{\pi_0 R/\pi_0 \mathbb{1}}, \pi_0 S), \quad t \geq s > 0,$$

where $\pi_0 S$ is given the $\pi_0 R$-module structure via ϕ.

Proof sketch.

This is the Bousfield-Kan spectral sequence applied to the tower $\{\mathrm{Map}_{\mathcal{M}_s}(R_{\leq s}, S_{\leq s})\}$.

□

The mapping space spectral sequence

Corollary

Let $\phi : R \to S$ be a morphism of E-local $\mathbb{E}_\infty$-rings. There is a first quadrant spectral sequence converging conditionally to $\pi_{t-s}(\mathrm{Map}_{\mathbb{E}_\infty}(R, S), \phi)$ with

$$E_1^{0,0} = \mathrm{Map}_{\mathcal{C}Alg(\mathcal{C}omod_{E_*E})}(E_*R, E_*S),$$
$$E_1^{s,t} = \mathrm{Ext}^{2s-t,s}_{\mathcal{M}od_{E_*R}(\mathcal{C}omod_{E_*E})}(\mathbb{L}_{E_*R/E_*}, E_*S), \quad t \geq s > 0,$$

where E_*S is given the E_*R-module structure via ϕ.

Primer on synthetic spectra

Goerss-Hopkins obstruction theory

Application to Morava E-theory

The main theorem

Recall our main goal:

Theorem (Goerss-Hopkins-Miller)

Let E be Lubin-Tate theory and $\mathbb{G}$ the Morava stabilizer group. Let $\mathcal{M}_{\mathbb{E}_\infty}(E)$ be the moduli space of $\mathbb{E}_\infty$-rings that are equivalent to E as homotopy associative rings. Then, $\mathcal{M}_{\mathbb{E}_\infty}(E) \simeq B\mathbb{G}$.

Proof.

- Instead of realizing E_* as an $\mathbb{E}_\infty$-ring, we realize the comodule algebra E_*E, i.e., find an E-local $\mathbb{E}_\infty$-ring A such that $E_*A \cong E_*E$. This is enough by a universal coefficient spectral sequence argument.

Proving the main theorem

Proof.

- Apply Goerss-Hopkins obstruction theory: the obstructions to existence and uniqueness of lifts live in

$$\mathrm{Ext}^{s,t}_{\mathcal{M}od_{E_*E}(\mathcal{C}omod_{E_*E})}(\mathbb{L}_{E_*E/E_*}, E_*E).$$

  We want to show these groups vanish unless $(s,t) = (0,0)$.

- The free-forget adjunction from modules to comodules induces an isomorphism

$$\mathrm{Ext}^{*,*}_{\mathcal{M}od_{E_*E}(\mathcal{C}omod_{E_*E})}(\mathbb{L}_{E_*E/E_*}, E_*E) \cong \mathrm{Ext}^{*,*}_{\mathcal{M}od_{E_*}}(\mathbb{L}_{E_*E/E_*}, E_*).$$

Proving the main theorem

Proof.

- Want to show $\mathrm{Ext}^{*,*}_{\mathcal{M}od_{E_*}}(\mathbb{L}_{E_*E/E_*}, E_*) \cong 0$.
- Filter the target E_* by powers of its maximal ideal $\mathfrak{m} = (p, u_1, \ldots, u_{h-1})$. This gives rise to a spectral sequence computing $\mathrm{Ext}^{*,*}_{\mathcal{M}od_{E_*}}(\mathbb{L}_{E_*E/E_*}, E_*)$ with E_2-term

$$\mathrm{Ext}^{p,*}_{\mathcal{M}od_{E_*/\mathfrak{m}}}(\mathbb{L}_{E_*E/E_*} \otimes^{\mathbb{L}}_{E_*} E_*/\mathfrak{m}, \mathfrak{m}^q/\mathfrak{m}^{q+1}).$$

- Suffice to show that $\mathbb{L}_{E_*E/E_*} \otimes^{\mathbb{L}}_{E_*} E_*/\mathfrak{m} \simeq 0$.

Proving the main theorem

Proof.

- Want to show $\mathbb{L}_{E_*E/E_*} \otimes^{\mathbb{L}}_{E_*} E_*/\mathfrak{m} \simeq 0$.
- E_*E is flat over E_*; by flat base change,

$$\mathbb{L}_{E_*E/E_*} \otimes^{\mathbb{L}}_{E_*} E_*/\mathfrak{m} \simeq \mathbb{L}_{(E_*E/\mathfrak{m})/(E_*/\mathfrak{m})} \simeq E_* \otimes_{E_0} \mathbb{L}_{(E_0E/\mathfrak{m})/(E_0/\mathfrak{m})}.$$

- $E_0/\mathfrak{m} \cong k$ is perfect, and so is $E_0E/\mathfrak{m} \cong \mathrm{Hom}_{\mathrm{cts}}(\mathbb{G}, k)$.

Claim

The cotangent complex of any morphism between perfect $\mathbb{F}_p$-algebras vanishes.

- So $\mathbb{L}_{(E_0E/\mathfrak{m})/(E_0/\mathfrak{m})} \simeq 0$.

The cotangent complex of perfect $\mathbb{F}_p$-algebras

Claim

The cotangent complex of any morphism between perfect $\mathbb{F}_p$-algebras vanishes.

Proof.

The Frobenius automorphism induces an isomorphism on cotangent complexes, but the map is given by

$$dx \mapsto d(x^p) = px^{p-1}\,dx = 0.$$

□

Concluding the main theorem

Proof.

- Therefore, all obstructions to existence and uniqueness of an $\mathbb{E}_\infty$-ring structure on E vanish.
- There is a unique $\mathbb{E}_\infty$-structure on E.
- Moreover,

$$\mathrm{Aut}_{\mathbb{E}_\infty}(E) \cong \mathrm{Aut}_{\mathcal{C}Alg(\mathcal{C}omod_{E_*E})}(E_*E) \cong \mathbb{G}.$$

□

Remark

In general, $\mathrm{Map}_{\mathbb{E}_\infty}(E(k_1, \Gamma_1), E(k_2, \Gamma_2))$ is homotopy discrete, with $\pi_0 = \mathrm{Hom}_{FGL}((k_1, \Gamma_1), (k_2, \Gamma_2))$.

The end

Thank you for listening!