The Landweber Exact Functor Theorem

Kevin Chang

Juvitop

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References

- *Topological Modular Forms*, Chapter 4 (Douglas, Francis, Henriques, Hill)
- COCTALOS, Lecture 20 (Hopkins)
- Lecture 15 (Lurie)

What is the LEFT?

- Recall: Complex oriented cohomology theories $\implies$ formal group laws.

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- Q: Can we go the other way? (i.e. FGLs $\implies$ spectra)
- A: It depends.

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- This automatically satisfies all the homology axioms except exactness.
- Flatness? Nah, too strong. $MU_* \cong \mathbb{Z}[t_n | n \in \mathbb{N}]$ is massive.
- Recall that $MU_*(X)$ is an (MU_*, MU_*MU)-comodule. Thus, it suffices to consider flatness wrt comodules.

The Landweber exact functor theorem

Definition

$[p]_{MU_*}(x) = \sum_{k=0}^{\infty} a_k x^{k+1}$. Define $v_i := a_{p^i-1}$ and $I_{p,n} := (p, v_1, \ldots, v_{n-1})$.

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An FGL F classified by $MU_* \to R$ is **Landweber exact** if $M \mapsto M \otimes_{MU_*} R$ is an exact functor from (MU_*, MU_*MU) -comodules to R-modules.

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LEFT (Classical)

Let $MU_* \to R$ classify a FGL F. If

$$v_n : R/I_{p,n} \to R/I_{p,n}$$

is injective for all p, n, then F is Landweber exact.

Proof sketch of LEFT (Classical)

LEFT (Classical)

$v_n : R/I_{p,n} \to R/I_{p,n}$ injective $\forall p, n \implies$ Landweber exact.

Theorem (Morava, Landweber)

The invariant prime ideals of MU_* are the $I_{p,n}$'s. (Invariant ideal = subcomodule of MU_*)

Landweber filtration theorem

Every coherent (MU_*, MU_*MU)-comodule M has a finite filtration whose subquotients are iso to $MU_*/I_{p,n}$.

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For coherent comodules, it suffices to prove

$$\mathrm{Tor}_1^{MU_*}(MU_*/I_{p,n}, R) = 0.$$

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Base case ($n = 0$):

$$0 \to \mathrm{Tor}_1^{MU_*}(MU_*/p, R) \to R \xrightarrow{p} R \to R/p \to 0$$

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Inductive step (assume $\mathrm{Tor}_1^{MU_*}(MU_*/I_{p,n}, R) = 0$):

$$0 \to \mathrm{Tor}_1^{MU_*}(MU_*/I_{p,n+1}, R) \to R/I_{p,n} \xrightarrow{v_n} R/I_{p,n} \to R/I_{p,n+1} \to 0$$

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Landweber's original proof (that we get a homology theory)

Note that $MU_*(X)$ is coherent for finite complexes X. Thus, $X \mapsto MU_*(X) \otimes_{MU_*} R$ is a homology theory.

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Or...

Theorem (Miller, Ravenel)

Every (MU_*, MU_*MU)-comodule is a union of coherent subcomodules.

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- $[p]_{F_a}(x) = px \implies v_n = 0 \; \forall n > 0$.
- Multiplication by v_1 is not injective on $\mathbb{Z}/p$, so F_a is not Landweber exact.
- We cannot get $H\mathbb{Z}$ from Landweber exactness.

Applications

Example

The additive formal group law over $\mathbb{Q}$ is Landweber exact (p is invertible):

$$MU_*(X) \otimes_{MU_*} \mathbb{Q} \cong H_*(X; \mathbb{Q}).$$

Example

The multiplicative formal group law over $\mathbb{Z}[\beta, \beta^{-1}]$ ($|\beta| = -2$):

$$F_m(x, y) = x + y + \beta xy$$
$$[p]_{F_m}(x) \equiv \beta^{p-1} x^p \pmod{p}.$$

We get K-theory from Landweber exactness (Todd genus):

$$MU_*(X) \otimes_{MU_*} \mathbb{Z}[\beta, \beta^{-1}] \cong K_*(X).$$

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A FGL over a torsion-free $\mathbb{Z}_{(p)}$-algebra is p-**typical** if its logarithm is of the form $\sum_i l_i x^{p^i}$. (Definition for non-torsion-free $\mathbb{Z}_{(p)}$-algebras is more complicated.)

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Theorem

(BP_*, BP_*BP) classifies p-typical FGLs and strict isos between p-typical FGLs.

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Q: When do FGs give us homology theories?

Landweber exactness and stacks

Definition

A morphism $\mathcal{F} : \mathcal{M} \to \mathcal{N}$ is **representable** if for all $\operatorname{Spec} A \to \mathcal{N}$, the pullback $\mathcal{M} \times_{\mathcal{N}} \operatorname{Spec} A$ is equivalent to an affine scheme $\operatorname{Spec} P$.

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Definition

A representable morphism is **flat** (resp. a **covering**) if all pullbacks to affine schemes are flat (resp. coverings).

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Proposition

To check representability/flatness/faithful flatness of $\mathcal{N} \to \mathcal{M}_{(A,\Gamma)}$, it is enough to check on the pullback by $\operatorname{Spec} A \to \mathcal{M}_{(A,\Gamma)}$.

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Proposition

$\mathcal{F} : \operatorname{Spec} R \to \mathcal{M}_{(A,\Gamma)}$ flat $\iff$
$\mathcal{F}^* : \operatorname{QCoh}(\mathcal{M}_{(A,\Gamma)}) \to \operatorname{QCoh}(\operatorname{Spec} R)$ exact.

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Proposition

$\mathcal{F} : \operatorname{Spec} R \to \mathcal{M}^s_{FG}$ flat $\iff$ $\mathcal{F}$ Landweber exact FG
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- Degree-2 FGLs with degree-0 FGLs.

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Corollary

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Corollary

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Remark

To prove this, need to show spectra are determined up to unique homotopy equivalence (no phantom maps, see Lurie Lecture 17).

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Example

Want Hopf algebroid representing Weierstrass equations and isos.

$$A := \mathbb{Z}[a_1, a_2, a_3, a_4, a_6, \Delta^{-1}]$$
$$\Gamma := A[u^{\pm 1}, r, s, t]$$
$$\mathcal{M}_{ell} := \mathcal{M}_{(A,\Gamma)}.$$

$\mathcal{M}_{ell}$ is the **moduli stack of elliptic curves**.

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$\exists$ presheaf on flat site of $\mathcal{M}_{ell}$ valued in Ho(Spectra).

- Can we do better than Ho(Spectra)?
- Q: Can we turn flat $\mathcal{N} \to \mathcal{M}_{FG}$ into a homology theory?