Obstruction Theory for (∞, n) - Categories

Outline.

(1) What we've been doing

(2) The contractibility of the space of framed functions

(3) Obstruction Theory for (∞, n) - categories

1 What we've been doing

Recall.

(1) We reduced the cobordism hypothesis to an 'inductive formulation that roughly says the (∞,n)-category Bord_n is obtained from the $(\infty,n-1)$-category Bord_{n-1} by freely adjoining an $O(n)$-equivariant n-morphism

$$\eta : \emptyset \longrightarrow S^{n-1}$$

(2) We used Morse theory to progressively add generators and relations

$$\begin{tikzcd}
\mathrm{Bord}_{n-1} \arrow[r] & F_0 \arrow[r] & F_1 \arrow[r] & \cdots \arrow[r] & \mathrm{Bord}_n^{ff} \arrow[d, "\text{forget}"] \\
 & & & & \mathrm{Bord}_n
\end{tikzcd}$$

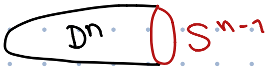
'framed functions'

Point. The cobordism hypothesis follows from the two claims.

← last time

Claim 1 (Cob. hypo., framed function version). $\mathcal{C}$ symm. mon. (∞,n)-category with duals, $Z_0 : \mathrm{Bord}_{n-1} \longrightarrow \mathcal{C}$ symm. mon. functor. There is an equivalence

$$\left\{ \begin{array}{c} \text{sym. mon. functors} \\ Z : \mathrm{Bord}_n^{ff} \longrightarrow \mathcal{C} \\ \text{extending } Z_0 \end{array} \right\} \xrightarrow{\sim} \left\{ \begin{array}{c} \text{nondegenerate} \\ O(n)\text{-equivariant} \\ n\text{-mors. } \eta : 1_{\mathcal{C}} \to Z_0(S^{n-1}) \end{array} \right\}$$

$Z \longmapsto Z\Big($  $\Big)$

Claim 2. The forgetful functor is an equivalence of (∞, n)-categories.

Today. A cohomology / obstruction theory approach to Claim 2.

2 The contractibility of the space of framed functions

Observation. Claim 2 is equivalent to the claim that given:

(1) An $(n-2)$-manifold M.

(2) $(n-1)$-manifolds X and X' and diffeomorphisms

$$\partial X \xrightarrow{\sim} M \xleftarrow{\sim} \partial X'$$

(3) A cobordism $B: X \to X'$ trivial along the common boundary M.

Then the space $\mathrm{Fun}^{fr}(B)$ of framed functions on B is contractible.

Theorem (Igusa 1987). With B as above, $\mathrm{Fun}^{fr}(B)$ is $(n-1)$-connected.

Theorem ($\overbrace{\text{Eliashberg-Mishachev 2011}}^{\text{using a lot of geometry/foliations}}$, $\overbrace{\text{Kupers after Galatius 2017}}^{\text{using a general h-principle}}$)
The space $\mathrm{Fun}^{fr}(B)$ is contractible.

Lurie's Method. Igusa's result shows that

$$\mathrm{Bord}_n^{ff} \longrightarrow \mathrm{Bord}_n$$

is an equiv. on homotopy $(n+1,n)$-categories. Use this + a 'cohomology theory for (∞,n)-categories'.

Review of the Classical Case

Note. For $n=0$, the input is an equivalence on fundamental groupoids.

Classical Theorem. A morphism $f: X \to Y$ in Spc is an equivalence iff:

(1) f induces an equivalence on fundamental groupoids

(2) For every integer $m \geq 0$ and local system

$$A: \tau_{\leq 1} Y \longrightarrow \mathrm{Ab},$$

the induced map

$$f^*: H^m(Y; A) \longrightarrow H^m(X; A)$$

is an isomorphism.

Motivation: Overview of the proof

Recall. Let $n \geq 0$. A space X is n-truncated if for all $x \in X$ and $i > n$, we have $\pi_i(X, x) = 0$.

> Write $\mathrm{Spc}_{\leq n} \subset \mathrm{Spc}$ for the full subcat. spanned by the n-truncated spaces.

> The inclusion $\mathrm{Spc}_{\leq n} \hookrightarrow \mathrm{Spc}$ admits a left adjoint $\tau_{\leq n}: \mathrm{Spc} \longrightarrow \mathrm{Spc}_{\leq n}$. Moreover, $\tau_{\leq n}$ preserves products.

Postnikov Convergence. For any space Z,

$$Z \xrightarrow{\sim} \lim \left(\cdots \to \tau_{\leq n} Z \to \cdots \to \tau_{\leq 1} Z \to \tau_{\leq 0} Z \right).$$

Even better,

$$\mathrm{Spc} \xrightarrow{\sim} \lim \left(\cdots \to \mathrm{Spc}_{\leq n+1} \xrightarrow{\tau_{\leq n}} \mathrm{Spc}_{\leq n} \to \cdots \right).$$

in $\mathrm{Cat}_{(\infty,1)}$

Point. In the theorem we want to show that for every space Z,

$$\begin{tikzcd} f^* : \mathrm{Map}(Y,Z) \arrow[r, "\sim"] \arrow[d, phantom, "\cong" rotate=90] & \mathrm{Map}(X,Z) \arrow[d, phantom, "\cong" rotate=90] \\ \lim\limits_n \mathrm{Map}(Y, \tau_{\leq n} Z) \arrow[r] & \lim\limits_n \mathrm{Map}(X, \tau_{\leq n} Z) \end{tikzcd}$$

$\rightsquigarrow$ we can reduce to the case Z is n-truncated

(base) $n=1$ is condition (1)

(induction) Want to use (2) to show that

$$\mathrm{Map}(Y, \tau_{\leq n-1} Z) \xrightarrow{\sim} \mathrm{Map}(X, \tau_{\leq n-1} Z)$$

$$\Downarrow$$

$$\mathrm{Map}(Y, \tau_{\leq n} Z) \xrightarrow{\sim} \mathrm{Map}(X, \tau_{\leq n} Z)$$

Observe. By the LES of a fibration, for each $z \in Z$,

$$\mathrm{fib}_z (\tau_{\leq n} Z \to \tau_{\leq n-1} Z) \text{ is a } K(A_{z}, n)$$

$z \mapsto A_z$ defines a local system $A : \tau_{\leq 1} Z \to \mathrm{Ab}$

Quick interlude on cohomology with local coefficients.

> Usual cohomology: Y space, A abelian group

$$H^m(Y;A) \simeq \pi_0 \mathrm{Map}(Y, K(A,m))$$

Sections of $K(A,n)\times Y \xrightarrow{pr_2} Y$

$$\simeq \pi_0 \mathrm{Map}_{Spc_{/Y}}(Y, K(A,m)\times Y)$$

> Variant: If $A : \tau_{\le 1} Y \longrightarrow Ab$ is a local system, there's a unique morphism $K_Y(A,m) \longrightarrow Y$ with fibs. $K(A_y, m)$ varying with $y \mapsto A_y$:

$$\mathrm{Fun}(Y, Spc) \xrightarrow{\sim} Spc_{/Y}$$

$$[y \mapsto K(A_y, m)] \longmapsto K_Y(A,m)$$

Then

$$\boxed{H^m(Y;A) \simeq \pi_0 \mathrm{Map}_{Spc_{/Y}}(Y, K_Y(A,m))}$$

Point. To understand $\pi_* \mathrm{Map}(Y, \tau_{\le n} Z)$, choose a basepoint $\bar{g} : Y \longrightarrow \tau_{\le n} Z$ and write

$$g : Y \xrightarrow{\bar{g}} \tau_{\le n} Z \longrightarrow \tau_{\le n-1} Z .$$

We have LESs in homotopy:

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] \\
\pi_{k+1}(\mathrm{Map}(Y, \tau_{\leq n-1}Z), g) \arrow[r, "\sim", "\text{induction}"'] \arrow[d] & \pi_{k+1}(\mathrm{Map}(X, \tau_{\leq n-1}Z), gf) \arrow[d] \\
H^{n-k}(Y; g^*A) \arrow[r, "\sim", "(2)"'] \arrow[d] & H^{n-k}(X; f^*g^*A) \arrow[d] \\
\pi_k(\mathrm{Map}(Y, \tau_{\leq n}Z), \bar{g}) \arrow[r, "\sim", "5.\text{-lemma}"'] \arrow[d] & \pi_k(\mathrm{Map}(X, \tau_{\leq n}Z), \bar{g}f) \arrow[d] \\
\pi_k(\mathrm{Map}(Y, \tau_{\leq n-1}Z), g) \arrow[r, "\sim", "\text{induct.}"'] \arrow[d] & \pi_k(\mathrm{Map}(X, \tau_{\leq n-1}Z), gf) \arrow[d] \\
H^{n+1-k}(Y; g^*A) \arrow[r, "\sim", "(2)"'] \arrow[d] & H^{n+1-k}(X; f^*g^*A) \arrow[d] \\
\vdots & \vdots
\end{tikzcd}$$

3 Obstruction Theory for (∞,n)-categories

Notes.

(1) We'll explain what we want of such a theory
No proofs!

(2) Ongoing work of

$\left[\begin{array}{l}\text{Yonatan Harpaz, Joost Nuiten, \& Matan Prasma}\\ \text{Chris Kapulkin \& Morgan Opie}\end{array}\right.$

provides such a theory.

What do we need?

(1) Truncations & Postnikov convergence

(2) Local Systems

(∞,n)-categories

(3) Eilenberg-MacLane Objects

Truncations

Recall. Let $m \geq n \geq 0$. An (m,n)-category is an (∞,n)-category where all morphisms above level m are trivial.

> E.g., $(m,0)$-category = m-truncated Space.

Fact. The inclusion $\mathrm{Cat}_{(m,n)} \hookrightarrow \mathrm{Cat}_{(\infty,n)}$ admits a left adjoint

$$\underbrace{\tau^{(\infty,n)}_{\le m}}_{\text{for } n=0\text{, this is } n\text{-truncation}} : \mathrm{Cat}_{(\infty,n)} \longrightarrow \mathrm{Cat}_{(m,n)}$$

> Informally

$$\mathrm{Obj}(\tau^{(\infty,n)}_{\le m} C) := \mathrm{Obj}(C)$$

$$\mathrm{Map}_{\tau^{(\infty,n)}_{\le m} C}(X,Y) := \tau^{(\infty,n-1)}_{\le m-1} \underbrace{\mathrm{Map}_C(X,Y)}_{(\infty,n-1)\text{-Cat}}$$

Example. By Igusa, $\tau^{(\infty,n)}_{\le n+1} \mathrm{Bord}^{ff}_n \xrightarrow{\sim} \tau^{(\infty,n)}_{\le n+1} \mathrm{Bord}_n$.

Postnikov Convergence. $C \xrightarrow{\sim} \lim_{m} \tau_{\le m} C$

Local Systems

Not really precise/complete

"Definition". Define local systems on an (∞,n)-category inductively on n as follows

(0) For $n=0$, $\mathrm{Loc}(C) := \mathrm{Fun}(C,\mathrm{Ab}) \simeq \mathrm{Fun}(\tau_{\le 1}C, \mathrm{Ab})$.

(n) For $n>0$, a local system on C consists of

(a) An assignment for all $x,y \in C$, a local system

$$A_{x,y} \in \mathrm{Loc}(\underbrace{\mathrm{Map}_C(x,y)}_{(\infty,n-1)\text{-Cat}}).$$

(b) For each triple $x,y,z \in C$, a 'composition' morphism

$$pr_1^*(A_{x,y}) \times pr_2^*(A_{y,z}) \xrightarrow{m_{x,y,z}} c^*(A_{x,z})$$

composition

in $\mathrm{Loc}(\mathrm{Map}_C(x,y) \times \mathrm{Map}_C(y,z))$

+ Associativity and unitality conditions

Example. $C = 0 \longrightarrow 1$

Abelian groups: $A_{0,0}$, $A_{0,1}$, $A_{1,1}$ not $A_{1,0}$ since $\mathrm{Map}(1,0) = \emptyset$

Actions:

$$A_{0,0} \times A_{0,1} \longrightarrow A_{0,1}$$

$$A_{0,1} \times A_{1,1} \longrightarrow A_{0,1}$$

$$\left.\begin{array}{l} A_{0,0} \times A_{0,0} \longrightarrow A_{0,0} \\ A_{1,1} \times A_{1,1} \longrightarrow A_{1,1} \end{array}\right\} \text{the group addition}$$

Eilenberg-MacLane Objects & Cohomology

Definition. Let C be an (∞,n)-category and A a local system on C. Define (∞,n)-categories

$$K_C(A,m) \xrightarrow{q_A} C, \qquad m \geq n$$

inductively on n as follows:

Eilenberg-MacLane space over C

(0) $n=0$: $K_C(A,m)$ from before.

(n) $n > 0$: $\mathrm{Obj}(K_C(A,m)) := \mathrm{Obj}(C)$

$$\mathrm{Map}_{K_C(A,m)}(x,y) := K_{\mathrm{Map}_C(x,y)}(A_{x,y}, m-1)$$

Composition: induced by the maps $m_{x,y,z}$.

$$\boxed{H^m(C;A) := \left\{ \begin{array}{c} \text{iso. classes of} \\ \text{sections of} \\ q_A : K_C(A,m) \to C \end{array} \right\}}$$

there's a natural 0-section $C \to K_C(A,m)$ and mult. $A \times A \to A$ making this an abelian group

Variant. If C is symmetric monoidal, we can consider muliplicative local systems A on C:

$\rightsquigarrow$ $K_C(A,m)$ inherits a natural symm. mon. structure

$$\boxed{H^m_{\otimes}(C;A) := \left\{ \begin{array}{c} \text{iso. classes of} \\ \otimes\text{-sections of} \\ q_A : K_C(A,m) \to C \end{array} \right\}}$$

⚠ The forgetful map $H^m_{\otimes}(C;A) \to H^m(C;A)$ is not generally an isomorphism.

when $n=0$, this is a variant of the 'classical Theorem' for maps of E_∞-monoids

Proposition. Let $f: D \to D'$ be a symm. monoidal functor between symm. mon. (∞, n)-categories. Then f is an equivalence iff:

(1) $\tau_{\leq n+1}(f): \tau_{\leq n+1}(D) \to \tau_{\leq n+1}(D')$ is an equivalence of $(n+1,n)$-categories.

(2) For every multiplicative local system A on D' and $m \geq 0$,

$$f^*: H^m_\otimes(D'; A) \longrightarrow H^m_\otimes(D; f^*A)$$

is an isomorphism.

What we want. To prove the cobordism hypothesis, we need to see that for all multiplicative local systems A on Bord_n,

$$\underbrace{H^m_\otimes(\mathrm{Bord}_n, \mathrm{Bord}_n^{ff}; A)}_{\substack{\text{Relative cohomology:} \\ \text{fits into usual LES}}} = 0$$

- We have a LES

$$\begin{tikzcd}
\cdots \arrow[r] & H^m_\otimes(\mathrm{Bord}_n, \mathrm{Bord}_n^{ff}; A) \arrow[r] & H^m_\otimes(\mathrm{Bord}_n, \mathrm{Bord}_{n-1}; A) \arrow[dll, "\theta_m"'] & \\
H^m_\otimes(\mathrm{Bord}_n^{ff}, \mathrm{Bord}_{n-1}; f^*A) \arrow[r] & H^{m+1}_\otimes(\mathrm{Bord}_n, \mathrm{Bord}_n^{ff}; A) \arrow[r] & \cdots &
\end{tikzcd}$$

$f: \mathrm{Bord}_n^{ff} \to \mathrm{Bord}_n$

$H^m_\otimes(\mathrm{Bord}_n, \mathrm{Bord}_n^{ff}; A) = 0$ $\Longleftrightarrow$ θ_m is an isomorphism

for all m for all m

- By the cobordism hypothesis for framed functions

$$H^m_\otimes(\mathrm{Bord}_n^{ff}, \mathrm{Bord}_{n-1}; f^*A) \overset{\text{def}}{=} \left\{ \begin{array}{c} \text{Sections of} \\ K_{\mathrm{Bord}_n^{ff}}(f^*A, m) \to \mathrm{Bord}_n^{ff} \\ \text{that restrict to zero} \\ \text{on } \mathrm{Bord}_n \end{array} \right\}$$

$$\cong \left\{ \begin{array}{c} O(n)\text{-equivariant} \\ n\text{-morphisms} \\ 1_{K(f^*A,m)} = \emptyset \longrightarrow S^{n-1} \\ \text{in } K_{\mathrm{Bord}_n^{ff}}(f^*A, m) \end{array} \right\}$$

- For each $x \in BO(n)$ and we can evaluate the local system $A \in \mathrm{Loc}(\mathrm{Bord}_n)$ on the n-morphism $\eta_x : \emptyset \longrightarrow S^{n-1}$ in Bord_n to obtain an abelian group $L_{A,x}$.

  Component of $D^n : \emptyset \to S^{n-1}$ at $x \in BO(n)$.

  - The assignment $x \mapsto L_{A,x}$ defines a local system L_A on $BO(n)$

- Sending a section $s : \mathrm{Bord}_n^{ff} \longrightarrow K_{\mathrm{Bord}_n^{ff}}(f^*A, m)$ to the section

$$\begin{aligned} BO(n) &\longrightarrow K_{BO(n)}(L_A, m-n) \\ x &\longmapsto s(\eta_x) \end{aligned}$$

defines an isomorphism

$$H^m_\otimes(\mathrm{Bord}^{ff}_n, \mathrm{Bord}_{n-1}; f^*A) \xrightarrow{\sim} H^{m-n}(BO(n); L_A).$$

> So the cobordism hypothesis is equivalent to:

Claim 3 (Cobordism hypothesis, infintesimal version). For all m, the canonical map

$$H^m_\otimes(\mathrm{Bord}_n, \mathrm{Bord}_{n-1}; A) \longrightarrow H^{m-n}(BO(n); L_A)$$

is an isomorphism.

Example. If A is constant, then

$$H^m_\otimes(\mathrm{Bord}_n, \mathrm{Bord}_{n-1}; A) = H^m(|\mathrm{Bord}_n|, |\mathrm{Bord}_{n-1}|; A)$$

classifying space

By the Galatius-Madsen-Tillmann-Weiss Theorem,

$$|\mathrm{Bord}_k| \simeq \Omega^\infty \Sigma^k \underbrace{MTO(k)}_{\text{Madsen-Tillmann spectrum}}.$$

The Galatius-Madsen-Tillmann-Weiss fiber sequence

$$\Sigma^{n-1} MTO(n-1) \longrightarrow \Sigma^n MTO(n) \longrightarrow \Sigma^{\infty+n}_+ BO(n)$$

implies Claim 3 in this case.