Cobordism Hypothesis talk:

We've reduced CH to $\mathrm{Bord}_1 \to \cdots \to \mathrm{Bord}_n$, now reduce this.

Lets start w/ length 2: $B_1 \xrightarrow{\text{ess surj}} B_2$ s.m. From 1-cat to a 2-cat.

Need to describe $\mathrm{Map}_{B_2}(F(X), F(Y))$.

a priori this is functorial in $B_1^{op} \times B_1$, but if we assume B_1 has duals then $\mathrm{Map}_{B_2}(F(X), F(Y)) = \mathrm{Map}_{B_2}(1, F(X^\vee \otimes Y))$ is functorial in B_1!

So ~~take~~ let $M(X) := \mathrm{Map}_{B_2}(1, F(X))$, then $X \mapsto M(X)$ is a functor $B_1 \to \mathrm{Cat}_{(\infty,1)}$

Detour: We can "unstraighten" the above functor
Grothendieck Construction:
Let B, M be as above, then

$\mathrm{Groth}(B, M)$ has objects (X, η), $X \in B$, $\eta \in M(X)$
Mapping space (X,η) to (X', η') a classifying space for data
(f, α), $f \in \mathrm{Map}_B(X, X')$, $\alpha \in \mathrm{Map}_{M(X')}(f_! \eta, \eta')$

Have a functor $\mathrm{Groth}(B,M) \to B$, $(X,\eta) \mapsto X$ s.t.
$X \in B$, $\mathrm{Groth}(B,M)_X \simeq M(X)$, & these fibers vary functorially in B, in the sense that for all $f: X \to Y$ in B, & $\bar{X} \in \mathrm{Groth}(B,M)_X$, $\exists\, \bar{Y} \in \mathrm{Groth}(B,M)_Y$ & $\bar{f}: \bar{X} \to \bar{Y}$ w/ is ess. uniquely determined by $\bar{X}$ & f

Prop 3.3.24: $M \mapsto \mathrm{Groth}(B,M)$ determines an equivalence between
a.) Functors $B \to \mathrm{Cat}_{(\infty,1)}$ (lax s.m.)
b.) coCart Fib's $\pi: \mathcal{C} \to B$. (s.m.)

Corollary (Prop 3.3.28): $B_1 \to B_2$ as above is equivalent to a s.m. coCart Fib $\mathcal{C} \to B_1$.

Now for higher n.

$B_1 \to B_2 \to \dots \to B_n$ ess surj, s.m. Functors, B_1 has duals
$F_i : B_1 \to B_i$
Let $M_i : B_1 \to \cancel{\mathcal{A}}\, Cat_{(\infty, i-1)}$, $M_i(x) := Map_{B_i}(1, F_i(x))$
More general version of the GC gives $\pi_i : \mathcal{C}_i \to B_1$ s.m. cofart Fib
w/ $\mathcal{C}_i$ an $(\infty, i-1)$-cat
& have ess surj s.m. Functors

$$\begin{tikzcd} \mathcal{C}_2 \arrow[r] \arrow[drr] & \mathcal{C}_3 \arrow[r] \arrow[dr] & \dots \arrow[r] \arrow[d] & \mathcal{C}_n \arrow[dl] \\ & & B_1 & \end{tikzcd}$$

IF $\mathcal{C}_2$ has duals then we've basically just reduced n by 1.

Prop 3.3.29: $B_1 \to B_2$ as before, TFAE:
1.) All maps $1 \to X$ in B_2 have left adjoints.
2.) The cat ~~$\mathcal{C}_F$~~ has duals Groth(B_1, F)

Now let $G_i : \mathcal{C}_2 \to \mathcal{C}_i$, & $N_i : \mathcal{C}_2 \to Cat_{(\infty, i-2)}$ $N_i(x) := Map_{\mathcal{C}_i}(1, G_i(x))$

Notice these maps are all relative to B_1 so they are determined by maps on the Fibers $Fib(\mathcal{C}_i \to B_1) = M_i(1) \simeq Map_{B_i}(1, F_i(1)) \simeq Map_{B_i}(1,1)$
$=: \Omega B_i$

So new seq $\Omega B_2 \to \Omega B_3 \to \dots \to \Omega B_n$

Goal: Keep going until $n=2$, then use previous result.

Def: A skeletal seq of length n is a diagram

$$B_1 \xrightarrow{F_1} B_2 \xrightarrow{F_2} \dots \xrightarrow{F_{n-1}} B_n,$$ such that

1.) B_k a s.m. k-cat
2.) F_k is $(k-1)$-connective
3.) B_1 has duals
4.) For $2 \le k \le n$ every map $\alpha : 1 \to X$ in $\Omega^{k-2} B_k$ has a LA.

We've ~~seen how to reduce~~

Prop 3.3.31: Fix a SS $F: B_1 \to B_2$ of len 2. Then the above discussion gives an equivalence

1.) SS's of len n $\quad B_1 \to B_2 \to \dots \to B_n$, beg w/ F
2.) SS's of len $n-1$ $\quad \Omega B_2 \to \mathcal{C}_2 \to \dots \to \mathcal{C}_{n-1}$

So we can reduce SS's to 1-cat data, but how do we organize all that data?

Def: A Categorical Chain Complex of length n consists of
a.) A seq of s.m. cocart Fibs $\{\mathcal{C}_k \to \mathcal{Z}_{k-1}\}_{1 \le k \le n}$, between 1-cat's & $\mathcal{Z}_0 \simeq *$ & each $\mathcal{C}_k$ has duals.
b.) $\mathcal{Z}_k \simeq \mathcal{C}_k \underset{\mathcal{Z}_{k-1}}{\times} \{1\}$

Construction: Given a SS $B_1 \to B_2 \to \dots \to B_n$, associate a C.C.C.

$$\mathcal{Z}_k = \Omega^{k-1} B_k$$
$$\mathcal{C}_1 = \mathcal{Z}_1, \ \& \ 2 \le k \le n \quad \mathcal{C}_n = \mathcal{C}[\Omega^{k-2} F_{k-1}]$$

ex: $\{B_1 \to *\}$
$\{\mathcal{C}[F] \to B\}$
$\{\mathcal{C}[\Omega F_2] \to \Omega B_2\}$

Finally: Applying all of this to $\mathrm{Bord}_1 \to \mathrm{Bord}_2 \to \dots \to \mathrm{Bord}_n$

get C.C.C. $\{\mathrm{Cob}_\partial^{un}(k) \to \mathrm{Cob}_t^{un}(k-1)\}_{1 \le k \le n}$

$\mathrm{Cob}_\partial^{un}(k)$ the $k-1$ Mflds w/ boundary, & Mapping spaces ~~objects~~ classifying spaces of bordisms

$\mathrm{Cob}_t^{un}(k-1)$ are closed $(k-2)$-Mflds.