Cobordism hypothesis seminar: Reduction to the unoriented case

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Recall (yesterday)

Let (B, ξ) — space, ξ rank n vector bundle on B w/ inner product. $\begin{matrix}\xi\\ \downarrow \\ B = BG\end{matrix}$ $G \xrightarrow{\chi} O(n)$

$\xi_0 = \{(b,v,w) : (b,v) \in B_0 : w \in \xi_b : \langle v,w\rangle = 0\}$.

$\downarrow$ rank $n-1$ vector bundle on B_0.

B_0 = sphere bundle of ξ on $B = \{(b,v) : b \in B,\ v \in \xi_b : |v| = 1\}$

Rmk This construction is also used to compute H^* of classifying spaces of [structured] vector bundles inductively.

Observations. projection $B_0 \xrightarrow{p} B$. $\to$ exhibits B_0 as S^{n-1}-bundle.

$p^*\xi \cong \xi_0 \oplus \mathbb{R}$. $\longrightarrow$ (B_0, ξ_0) universal w/ this property.

$\rightsquigarrow$ gives functor $\mathrm{Bord}_{n-1}^{(B_0,\xi_0)} \longrightarrow \mathrm{Bord}_n^{(B,\xi)}$.

(*) Thm (CH, inductive, Lurie 3.1.8). Let $\mathcal{C}$ symmetric monoidal (∞,n)-category w/ duals, and let $Z_0 : \mathrm{Bord}_{n-1}^{(B_0,\xi_0)} \longrightarrow \mathcal{C}$ a symmetric monoidal functor.

Then the following data are equivalent:

(1) symmetric monoidal ("$\otimes$") functors. $Z : \mathrm{Bord}_n^{(B,\xi)} \longrightarrow \mathcal{C}$.

(2) families of nondegenerate n-morphisms. $\eta_b : \mathbb{1}_{\mathcal{C}} \longrightarrow Z_0(S^{\xi_b})$. parametrized by $b \in B$.

§ Introduction

Fix notation: $\mathcal{C}$ symmetric monoidal (∞,n)-category w/ duals.

"$\otimes$ —" = "symmetric monoidal —"

In the following, "topological space" = ∞-groupoid / homotopy type.

Reference: Lurie's Cobordism hypothesis, § 3.2

Disclaimer/remark: all mistakes are my own/use these notes at your own risk.

I'm happy to chat about the contents during a problem session/let me know if there are mistakes I should fix!

Observation: For any continuous group homomorphism $\chi : G \to O(n)$, have a forgetful functor.

$\mathrm{Bord}_n^G \longrightarrow \mathrm{Bord}_n$.

> Bord_n is the "largest" bordism category

Goal: use this forgetful functor to deduce the cobordism hypothesis for G-manifolds/(mflds w/ (B,ξ)-structure) from the cobordism hypothesis for unoriented manifolds.

Idea "adjunction formula" 3.2.7/8

Let $Z : \mathrm{Bord}_n^G \longrightarrow \mathcal{C}$ $\otimes$-functor & $M \in \mathrm{Bord}_n$

$\left(\text{"}=\text{"}\ \mathrm{Bord}_n^{(B,\xi)}\right)$.

Consider. $\mathrm{Bord}_n \rightsquigarrow$ "maps from spaces/∞-grpds to $\mathcal{C}$".

$M \longmapsto B_G M$ = classifying space for G-structures on M. $\to \mathcal{C}$.

choice of lift of M to Bord_n^G — $\underbrace{x \in B_G M \longmapsto Z(M,x)}_{\text{loc sys. on } B_G M \text{ w/ values in } \mathcal{C}}$.

Idea: regard collection $\{B_G M,\ B_G M \to \mathcal{C}\}_{M \in \mathrm{Bord}_n}$ as an (∞,n)-category $\mathrm{Fam}_n\,\mathcal{C}$.

> Then $M \longmapsto \{Z(M,x)\}_{x \in B_G M}$. determines a $\otimes$-functor

$\tilde{Z} : \mathrm{Bord}_n \longrightarrow \mathrm{Fam}_n\,\mathcal{C}$.

> Show Z can be recovered from $\tilde{Z}$.

§ Setup

Def (inductive) Let $\mathcal{C}$ an (∞,n)-category. Define an (∞,n)-category $\mathrm{Fam}_n(\mathcal{C})$

$(X, f : X \to \mathcal{C})$ where X space/∞-grpd, & f functor.

§ Setup

Def (Inductive) Let $\mathcal{C}$ an (∞,n)-category. Define an (∞,n)-category $\mathrm{Fam}_n(\mathcal{C})$

(1) Objects: pairs $(X, f: X \to \mathcal{C})$ where X space/∞-grpd., & f functor.

(2) $(n=0)$: morphism $(X \xrightarrow{f} \mathcal{C}) \to (Y \xrightarrow{g} \mathcal{C})$ is a map of spaces

$$\begin{tikzcd} X \arrow[rr, "h"] \arrow[dr, "f"'] & \arrow[d, phantom, "\Rightarrow"] & Y \arrow[dl, "g"] \\ & \mathcal{C} & \end{tikzcd}$$

& an equivalence of double composite.

(3) $(n>0)$: $(x,y) \in X \times Y$. $\mathcal{F}_{x,y} := \mathrm{Fam}_{n-1} \mathrm{Map}_{\mathcal{C}}(f(x), g(y))$. $\rightsquigarrow$ local system on $X \times Y$.

$\mathrm{Map}_{\mathrm{Fam}_n(\mathcal{C})}((X \xrightarrow{f} \mathcal{C}), (Y \xrightarrow{g} \mathcal{C}))$ = global sections of $\mathcal{F}$. $\leftarrow$ $(\infty, n-1)$-category

Ex $\mathcal{C} = *$ $n=1$ $\mathrm{Fam}_1(*) = \mathrm{Fam}_1$

- objects: ∞-grpds / spaces X
- morphisms: correspondences, i.e. a morphism $X \rightsquigarrow Y$ in Fam_1 is a space Z with maps. $\begin{tikzcd}[column sep=small] & Z \arrow[dl] \arrow[dr] & \\ X & & Y \end{tikzcd}$ $\longleftrightarrow$ $Z \to X \times Y$.

Why? For any $(x,y) \in X \times Y$, $\mathcal{F}_{x,y} = \mathrm{Fam}_0\left(\mathrm{Map}_*(f(x), g(y))\right) = \mathrm{Fam}_0 \mathrm{Map}_*(*,*) = \mathrm{Fam}_0(*)$

$\therefore$ "total space" of $\mathrm{Map}_*(*,*) = X \times Y$. $\overset{\mathrm{Fam}_0}{\rightsquigarrow}$ $Z \longrightarrow X \times Y$.

Rmk If $\mathcal{C}$ symmetric monoidal (∞,n) category, Then $\mathrm{Fam}_n(\mathcal{C})$ inherits a symmetric monoidal structure:

on objects: $(X \xrightarrow{f} \mathcal{C}) \otimes (Y \xrightarrow{g} \mathcal{C}) = (X \times Y \longrightarrow \mathcal{C})$
$(x,y) \mapsto f(x) \otimes g(y)$.

Claim If $\mathcal{C}$ has duals, so does $\mathrm{Fam}_n \mathcal{C}$.

Ex Every $X \in \mathrm{Fam}_1$ is dualizable: $X \xrightarrow{\Delta} X \times X$ $\quad X \longrightarrow * = \mathbb{1}$.

give correspondences $\mathbb{1} \overset{\mathrm{coev}}{\rightsquigarrow} X \times X$ $\quad X \times X \overset{\mathrm{ev}}{\rightsquigarrow} \mathbb{1}$.

$\therefore$ $X^\vee \simeq X$ canonically.

Now I can almost finally make precise the idea I indicated in the introduction:

Variant Ask for all spaces (objects, correspondences) in Fam_n to be pointed. $\rightsquigarrow \mathrm{Fam}_n^*$.

Define $\mathrm{Fam}_n^*(\mathcal{C}) := \mathrm{Fam}(\mathcal{C}) \times_{\mathrm{Fam}_n} \mathrm{Fam}_n^*$ "forget the basepoint"

Note There is an 'evaluation functor' $\mathrm{Fam}_n^*(\mathcal{C}) \xrightarrow{\mathrm{ev}} \mathcal{C}$
$(X \xrightarrow{f} \mathcal{C}, * \in X) \longmapsto f(*)$

Def Given (B, ξ) $\left(\overset{\text{can think}}{=} (BG, \xi)\right)$ define a functor

$Z_{(B,\xi)}: \mathrm{Bord}_n \longrightarrow \mathrm{Fam}_n$
$M \longmapsto B_{(B,\xi)} M$ = classifying space of (B,ξ)-structures on M.

Prop. Let B a topological space, ξ n-dim'l vector bundle on B w/ inner product. Then there is a homotopy pullback diagram of symmetric monoidal (∞,n) categories:

$$\begin{tikzcd} \mathrm{Bord}_n^{(B,\xi)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Fam}_n^* \arrow[d] \\ \mathrm{Bord}_n \arrow[r, "Z_{(B,\xi)}"] & \mathrm{Fam}_n. \end{tikzcd}$$

Idea "fiber" of LHS at $M^{\in}$ = space of (B,ξ)-structures on M = $B_{(B,\xi)}(M)$

by definition $= Z_{(B,\xi)}(M)$ = "fiber" of RHS over $Z_{(B,\xi)}(M) \in \mathrm{Fam}_n$.

Why? Unwind definitions: A k-morphism of $\mathrm{Bord}_n \times_{\mathrm{Fam}_n} \mathrm{Fam}_n^*$ is:
M k-morphism (i.e. k-manifold) of Bord_n
point of $Z_{(B,\xi)}(M)$
$\longleftrightarrow$ (B,ξ)-structure on M.

[illegible] to integrate into this picture.

Why? Unwind definitions ...

M k-morphism (i.e. k-manifold) of $\underline{\mathrm{Bord}}_n$ — point of $Z_{(B,\xi)}(M)$ $\leftrightarrow$ (B,ξ)-structure on M.

There's one last piece of the puzzle to integrate into this picture.

Q. What if we replace $\underline{\mathrm{Fam}}_n$ by $\underline{\mathrm{Fam}}_n(\mathcal{C})$?

Prop. Let $B, \mathcal{C}$ symmetric monoidal (∞,n)-categories & $Z : B \to \underline{\mathrm{Fam}}_n$ a $\otimes$-functor. Then the following are equivalent (as types of data):

(1) $\otimes$ functors $\widetilde{Z} : B \longrightarrow \underline{\mathrm{Fam}}_n(\mathcal{C})$ lifting Z.

(2) $\otimes$ functors $\widetilde{Z}' : B^* := B \times_{\underline{\mathrm{Fam}}_n} \underline{\mathrm{Fam}}_n^* \longrightarrow \mathcal{C}$.

The equivalence / assignment (1) $\longmapsto$ (2) is given by

$$B \times_{\underline{\mathrm{Fam}}_n} \underline{\mathrm{Fam}}_n^* \xrightarrow{\widetilde{Z} \times \mathrm{id}} \underline{\mathrm{Fam}}_n(\mathcal{C}) \times_{\underline{\mathrm{Fam}}_n} \underline{\mathrm{Fam}}_n^* = \underline{\mathrm{Fam}}_n^*(\mathcal{C}) \xrightarrow{\mathrm{ev}} \mathcal{C}.$$

Idea / Think "adjunction" / straightening-unstraightening equivalence in $\underline{\mathrm{Top}}^{\mathrm{s.g.}}$ / $\underline{\mathrm{Set}}$.

$$\mathrm{Map}(B, \mathrm{Map}(X, \mathcal{C})) \xrightarrow{\simeq} \mathrm{Map}(B \times X, \mathcal{C}).$$

Now we apply this with $B := \underline{\mathrm{Bord}}_n$, $Z := Z_{(B,\xi)}$ $\Rightarrow$ $B^* = \underline{\mathrm{Bord}}_n^{(B,\xi)}$

(*) **Prop.** Let $\mathcal{C}$ symmetric monoidal (∞,n) category. B a space, ξ $\mathbb{R}$ vector bundle on B of rank n, w/ inner product. Then the following data are equivalent:

(1) Symmetric monoidal functors $\underline{\mathrm{Bord}}_n \xrightarrow{\widetilde{Z}} \underline{\mathrm{Fam}}_n(\mathcal{C})$ lifting $Z_{(B,\xi)}$.

(2) symmetric monoidal functors $\widetilde{Z}' : \underline{\mathrm{Bord}}_n^{(B,\xi)} \to \mathcal{C}$.

§ Reduction to the unoriented case. Pf sketch:

(a) data of $\otimes$ functor $\widetilde{Z} : \underline{\mathrm{Bord}}_n^{(B,\xi)} \to \mathcal{C}$

$\overset{(*)}{\longleftrightarrow}$ $\otimes$ functor. $\widetilde{Z}' : \underline{\mathrm{Bord}}_n \longrightarrow \underline{\mathrm{Fam}}_n \mathcal{C}$ lifting $Z_{(B,\xi)}$.

(b) Recall: $B_0 :=$ sphere bundle of ξ on B.

$= \{ (b,v) : b \in B, \ \& \ v \in \xi_b \ \& \ |v| = 1 \}$.

B_0 has a $(n-1)$-dim'l vector bundle. $\xi_0 := \{ (b,v,w) : (b,v) \in B_0 \ \& \ w \in \xi_b \ \& \ \langle v,w \rangle \}$.

& $B_0 \xrightarrow{p} B$. $\qquad p^*\xi \cong \xi_0 \oplus \underline{\mathbb{R}}$ — (B_0,ξ_0) has universal property.

Observe There is a commutative diagram

$$\begin{tikzcd}
\underline{\mathrm{Bord}}_{n-1} \arrow[r] \arrow[d, "Z_{(B_0,\xi_0)}"'] & \underline{\mathrm{Bord}}_n \arrow[d, "Z_{(B,\xi)}"] \\
\underline{\mathrm{Fam}}_{n-1} \arrow[r] & \underline{\mathrm{Fam}}_n .
\end{tikzcd}$$

Apply inductive formulation (*)

[data of a symmetric monoidal functor $\widetilde{Z} : \underline{\mathrm{Bord}}_n \longrightarrow \underline{\mathrm{Fam}}_n(\mathcal{C})$ lifting $Z_{(B,\xi)}$] is equivalent to.

(i) $\otimes$ functor. $\widetilde{Z}_0 : \underline{\mathrm{Bord}}_{n-1} \longrightarrow \underline{\mathrm{Fam}}_n(\mathcal{C})$ lifting $Z_{(B_0,\xi_0)}$.

$\to$ (ii) For each $y \in BO(n)$, a nondegenerate n-morphism $\widetilde{\eta}_y : \mathbb{1} \longrightarrow \widetilde{Z}_0(S^{\zeta_y})$ in $\underline{\mathrm{Fam}}_n(\mathcal{C})$ lifting. $\eta_y : \mathbb{1} \longrightarrow Z(S^{\zeta_y})$ in $\underline{\mathrm{Fam}}_n$. $\qquad \widetilde{Z}_0(D^{\zeta_y})$.

ζ is universal bundle on $BO(n)$

(c) "adjunction" (*) $\Rightarrow$ (i) is equivalent to.

(i') symmetric monoidal functor. $\underline{\mathrm{Bord}}_{n-1}^{(B_0,\xi_0)} \longrightarrow \mathcal{C}$.

Observe (ii) almost looks like the extra data needed to extend a functor

$$\begin{tikzcd}
\underline{\mathrm{Bord}}_{n-1}^{(B,\xi_0)} \arrow[r] \arrow[d] & \mathcal{C} \\
\underline{\mathrm{Bord}}_n^{(B,\xi)} \arrow[ur, dashed] &
\end{tikzcd}$$

(d) Let $B' := \{ (x,y,\alpha) : x \in B, \ y \in BO(n), \ \alpha : \xi_x \xrightarrow{\sim} \zeta_y \text{ isometry} \}$.

$$\begin{tikzcd}
& B' \arrow[dl, "\phi"'] \arrow[dr] & \\
BO(n) & & B
\end{tikzcd}$$

$(x,y,\alpha) \mapsto y$, $\qquad (x,y,\alpha) \mapsto x$.

For $y \in BO(n)$, note that $B'_y \simeq B_{(B,\xi)} D^{\zeta_y}$ — classifying space for (B,ξ)-structures on D^{ζ_y}.

$\simeq Z_{(B,\xi)}(D^{\zeta_y})$

For $y \in BO(n)$, note that

$$\begin{aligned} B'_y &\simeq B_{(B,\xi)} D^{\xi_y} \\ &\simeq Z_{(B,\xi)}(D^{\xi_y}) \\ &\simeq \eta_y . \end{aligned}$$

Note that (ii) $\Longleftrightarrow$ For each $y \in BO(n)$, a nondegenerate n-morphism $\tilde{\eta}_y : \mathbb{1} \to \tilde{Z}_0(S^{\xi_y})$ in $\underbrace{\mathrm{Fam}_n}_{\text{unpack}} \mathcal{C}$ lifting $\eta_y : \mathbb{1} \to Z(S^{\xi_y})$.

unpack. ← i.e. use "unstraightening"

$\Updownarrow$

For all $y \in BO(n)$ & all $y' \in B'_y$, a nondegenerate n-morphism $\eta_{y'} : \mathbb{1} \longrightarrow \tilde{Z}'_0(S^{\xi_{\phi(y')}})$.

$\Updownarrow$

(ii') For all $y' \in B'$, a nondegenerate morphism.

(e). Since for all $x \in B$, $\mathrm{fib}_x(B' \longrightarrow B) \simeq \{(y, \alpha) : y \in BO(n),\ \alpha : \xi_y \cong \xi_x\}$
$\simeq EO(n) \simeq *$.

$\therefore$ $B' \xrightarrow{\simeq} B$ is htpy equivalence.

$\Longrightarrow$ (ii'') is equivalent to for all $b \in B$, a nondegenerate n-morphism $\eta_b : \mathbb{1} \to \tilde{Z}'_0(S^{\xi_b})$

<u>Rmk</u> Steps (d) & (e) are another application of the unstraightening ("adjunction") idea:

(d) show that:

(ii') a morphism $BO(n) \longrightarrow \mathrm{Fam}_n(\mathcal{C})$, $y \longmapsto$ (a morphism $B_{(B,\xi)} D^{\xi_y} \longrightarrow \mathcal{C}$)

is equivalent to

(ii') a morphism $\overbrace{\coprod_{y \in BO(n)} B_{(B,\xi)} D^{\xi_y}}^{\text{topologized appropriately}} \longrightarrow \mathcal{C}$

$$\begin{array}{c} \coprod_{y \in BO(n)} B_{(B,\xi)} D^{\xi_y} \\ \downarrow \\ BO(n) \end{array}$$

(y + other data) $\mapsto$ y

"unstraighten"

(e) the total space is homotopy equivalent to B, hence data of (ii') is equivalent to <u>data of (ii'')</u>.