

Recall (yesterday)

Let (B, ξ) w/ inner product.
 space rank n vector bundle on B .

$$\begin{array}{c} \xi \\ \downarrow \\ B = BG \end{array} \quad G \xrightarrow{\chi} O(n)$$

$$\xi_0 = \{(b, v, w) : (b, v) \in B_0 : w \in \xi_b : \langle v, w \rangle \geq 0\}$$

\downarrow rank $n-1$ vector bundle on B_0 .

$$B_0 = \text{sphere bundle of } \xi \text{ on } B = \{(b, v) : b \in B, v \in \xi_b : |v| = 1\}$$

Remark This construction is also used to compute H^* of classifying spaces of [structured] vector bundles inductively.

Observations: projection $B_0 \rightarrow B$ exhibits B_0 as S^{n-1} -bundle.

$$p^* \xi \cong \xi_0 \oplus \mathbb{R} \quad \text{--- } (B_0, \xi_0) \text{ universal w/ this property.}$$

\rightarrow gives functor $\text{Bord}_{n-1}^{(B_0, \xi_0)} \rightarrow \text{Bord}_n^{(B, \xi)}$.

Thm (C.H. Inductive, Lurie 3.1.8). Let \mathcal{C} symmetric monoidal (ω, n) -category w/ duals, and let $Z_0: \text{Bord}_{n-1}^{(B_0, \xi_0)} \rightarrow \mathcal{C}$ a symmetric monoidal functor.

Then the following data are equivalent:

- (1) symmetric monoidal (" \otimes ") functors $Z: \text{Bord}_n^{(B, \xi)} \rightarrow \mathcal{C}$.
- (2) families of nondegenerate n -morphisms $\eta_b: \mathbb{1}_{\mathcal{C}} \rightarrow Z_0(S_b^{\xi_b})$ parametrized by $b \in B$.

§ Introduction

Fix notation: \mathcal{C} symmetric monoidal (ω, n) -category w/ duals.

" \otimes ---" = "symmetric monoidal ---"

In the following, "topological space" = ω -groupoid / homotopy type.

Reference: Lurie's Cobordism hypothesis, § 3.2

Disclaimer/remark: all mistakes are my own/use these notes at your own risk.

I'm happy to chat about the contents during a problem session/let me know if there are mistakes I should fix!

Observation: For any continuous group homomorphism $\chi: G \rightarrow O(n)$, have a forgetful functor.

$$\text{Bord}_n^G \rightarrow \text{Bord}_n$$

Bord_n is the "largest" bordism category.

Goal: use this forgetful functor to deduce the cobordism hypothesis for G -manifolds from the cobordism hypothesis for unoriented manifolds. (mflds w/ (B, ξ) -structure).

Idea "adjunction formula" 3.2.7-18

$$\text{Let } Z: \text{Bord}_n^G \rightarrow \mathcal{C} \text{ } \otimes \text{ functor } \& \ M \in \text{Bord}_n$$

Consider.

$$\begin{array}{ccc} \text{Bord}_n & \rightsquigarrow & \text{"maps from spaces / } \omega\text{-grps to } \mathcal{C}\text{"} \\ M & \longmapsto & B_G M = \text{classifying space for } G\text{-structures on } M \rightarrow \mathcal{C} \end{array}$$

$$\begin{array}{ccc} & \xrightarrow{\omega} & \\ & X & \longmapsto Z(M, \chi) \\ & \underbrace{\hspace{10em}} & \\ & \text{loc sys. on } B_G M \text{ w/ values in } \mathcal{C}. & \end{array}$$

Idea: regard collection $\{B_G M, B_G M \rightarrow \mathcal{C}\}_{M \in \text{Bord}_n}$ as an (ω, n) -category $\text{Fam}_n \mathcal{C}$.

\rightarrow Then $M \longmapsto \{Z(M, \chi)\}_{\chi \in B_G M}$ determines a \otimes -functor

$$\tilde{Z}: \text{Bord}_n \rightarrow \text{Fam}_n \mathcal{C}$$

\rightarrow Show Z_i can be recovered from \tilde{Z} .

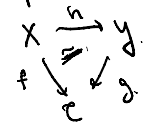
§ Setup

Def (Inductive) Let \mathcal{C} an (ω, n) -category. Define an (ω, n) -category $\text{Fam}_n(\mathcal{C})$... $(v, \xi, v \rightarrow \mathcal{C})$ where X space/ ω -prod. & f functor.

§ Setup

Def (inductive) Let \mathcal{C} an (∞, n) -category. Define an (∞, n) -category $\text{Fam}_n(\mathcal{C})$

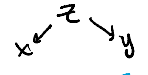
- (1) Objects: pairs $(X, f: X \rightarrow \mathcal{C})$ where X space (∞ -grpd), & f functor.
- (2) ($n=0$): morphism $(X \xrightarrow{f} \mathcal{C}) \rightarrow (Y \xrightarrow{g} \mathcal{C})$ is a map of spaces & an equivalence of double composites.



- (3) ($n>0$): $(x, y) \in X \times Y$. $\mathcal{F}_{x,y} := \text{Fam}_{n-1} \text{Map}_{\mathcal{C}}(f(x), g(y))$. \rightsquigarrow local system on $X \times Y$.
- $\text{Map}_{\text{Fam}_n(\mathcal{C})}((X \xrightarrow{f} \mathcal{C}), (Y \xrightarrow{g} \mathcal{C})) = \text{global sections of } \mathcal{F} \rightsquigarrow (\infty, n-1)\text{-category}$

Ex $\mathcal{C} = *$ $n=1$ $\text{Fam}_1(*) = \text{Fam}_1$

- objects: ∞ -grpds / spaces X
- morphisms: correspondences, i.e. a morphism $X \rightsquigarrow Y$ in Fam_1 is a space Z with maps $Z \rightarrow X \times Y$.



why? For any $(x, y) \in X \times Y$. $\mathcal{F}_{x,y} = \text{Fam}_0(\text{Map}_*(f(x), g(y))) = \text{Fam}_0 \text{Map}_*(*, *) = \text{Fam}_0(*)$
 \therefore "total space" of $\text{Map}_*(*, *) = X \times Y$. $\text{Fam}_0 Z \rightarrow X \times Y$.

Prop If \mathcal{C} symmetric monoidal (∞, n) category, then $\text{Fam}_n(\mathcal{C})$ inherits a symmetric monoidal structure:
 on objects: $(X \xrightarrow{f} \mathcal{C}) \otimes (Y \xrightarrow{g} \mathcal{C}) = (X \times Y \rightarrow \mathcal{C})$
 $(x, y) \mapsto f(x) \otimes g(y)$.

Claim if \mathcal{C} has duals, so does $\text{Fam}_n \mathcal{C}$.

Ex Every $X \in \text{Fam}_1$ is dualizable: $X \xrightarrow{\Delta} X \times X$ $X \rightarrow * = \mathbb{1}$
 give correspondences $\mathbb{1} \xrightarrow{\text{coev}} X \times X$ $X \times X \xrightarrow{\text{ev}} \mathbb{1}$.
 $\therefore X^\vee \in X$ canonically.

Now I can almost finally make precise the idea I indicated in the introduction:
Variant Ask for all spaces (objects, correspondences) in Fam_n to be pointed, $\rightsquigarrow \text{Fam}_n^*$.

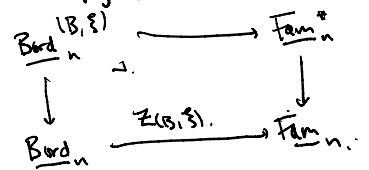
Define $\text{Fam}_n^*(\mathcal{C}) := \text{Fam}(\mathcal{C}) \times_{\text{Fam}_n} \text{Fam}_n^*$ "forget the basepoint"

Note There is an evaluation functor $\text{Fam}_n^*(\mathcal{C}) \xrightarrow{\text{ev}} \mathcal{C}$
 $(X \xrightarrow{f} \mathcal{C}, * \in X) \mapsto f(*)$

Def Given $(B, \xi) \stackrel{\text{can think}}{=} (B\mathcal{G}, \xi)$ define a functor

$Z_{(B, \xi)}: \text{Bord}_n \rightarrow \text{Fam}_n^*$
 $M \longmapsto B_{(B, \xi)} M = \text{classifying space of } (B, \xi)\text{-structures on } M$

Prop. Let B a topological space, ξ n -dim'l vector bundle on B w/ inner product.
 Then there is a homotopy pullback diagram of symmetric monoidal (∞, n) categories:



Idea "fiber" of LHS at $M \in \text{Bord}_n$ = space of (B, ξ) -structures on $M = B_{(B, \xi)}(M)$
 by definition = $Z_{(B, \xi)}(M) =$ "fiber" of RHS over $Z_{(B, \xi)}(M) \in \text{Fam}_n$.

Why? Unwind definitions: A k -morphism of $\text{Bord}_n \times_{\text{Fam}_n} \text{Fam}_n^*$ is:
 M k -morphism (i.e. k -manifold) of Bord_n \rightsquigarrow point of $Z_{(B, \xi)}(M)$ $\leftrightarrow (B, \xi)$ -structure on M .

... iterate into this picture.

Why? Unwind definitions: M k -manifold (i.e. k -manifold) of Bord_n

point of $Z(B, \xi)$ $\leftrightarrow (B, \xi)$ -structure on M .

There's one last piece of the puzzle to integrate into this picture.

Q What if we replace Fam_n by $\text{Fam}_n(C)$?

Prop. Let B, C , symmetric monoidal (op) categories & $Z: B \rightarrow \text{Fam}_n$ a \otimes -functor.

Then the following are equivalent (as types of data):

- (1) \otimes functors. $\bar{Z}: B \rightarrow \text{Fam}_n(C)$ lifting Z .
- (2) \otimes functors. $\bar{Z}': B^* := B \times_{\text{Fam}_n} \text{Fam}_n^* \rightarrow C$.

The equivalence/assignment (1) \leftrightarrow (2) is given by:

$$B \times_{\text{Fam}_n} \text{Fam}_n^* \xrightarrow{\bar{Z} \circ \text{id}} \text{Fam}_n(C) \times_{\text{Fam}_n} \text{Fam}_n^* = \text{Fam}_n^*(C) \xrightarrow{\text{or}} C.$$

Idea/Think "adjunction"/straightening-unstraightening equivalence in $\text{Top}^{\text{op}}/\text{Set}$
 $\text{Map}(B, \text{Map}(X, C)) \xrightarrow{=} \text{Map}(B \times X, C)$

Now we apply this with $B := \text{Bord}_n$ $Z := Z(B, \xi) \Rightarrow B^* = \text{Bord}_n(B, \xi)$

(*) Prop. Let C symmetric monoidal (op) catg. B a space, ξ \mathbb{R} vector bundle on B .

- (1) Symmetric monoidal functors $\text{Bord}_n \xrightarrow{\bar{Z}} \text{Fam}_n(C)$ lifting $Z(B, \xi)$.
- (2) Symmetric monoidal functors $\bar{Z}': \text{Bord}_n(B, \xi) \rightarrow C$.

§ Reduction to the unoriented case pf sketch:

- (a) data of \otimes functor $\bar{Z}: \text{Bord}_n(B, \xi) \rightarrow C$
 $\xleftrightarrow{(*)} \otimes$ functor. $\bar{Z}': \text{Bord}_n \rightarrow \text{Fam}_n(C)$ lifting $Z(B, \xi)$.

(b) Recall: $B_0 :=$ sphere bundle of ξ on B .

$$= \{(b, v) \mid b \in B, v \in \xi_b \text{ and } |v|=1\}.$$

B_0 has a $(n-1)$ -dim'd vector bundle. $\xi_0 := \{(b, v, w) \mid (b, v) \in B_0 \text{ and } w \in \xi_b \text{ and } \langle v, w \rangle = 0\}$.

$$\text{and } B_0 \xrightarrow{p} B. \quad p^* \xi \cong \xi_0 \oplus \mathbb{R} \quad \text{--- } (B_0, \xi_0) \text{ universal property.}$$

Observe There is a commutative diagram

$$\begin{array}{ccc} \text{Bord}_{n-1} & \longrightarrow & \text{Bord}_n \\ \downarrow Z(B_0, \xi_0) & & \downarrow Z(B, \xi) \\ \text{Fam}_{n-1} & \longrightarrow & \text{Fam}_n \end{array}$$

Apply inductive formulation: (*)

[data of a symmetric monoidal functor $Z: \text{Bord}_n \rightarrow \text{Fam}_n(C)$ lifting $Z(B, \xi)$] is equivalent to:

- (i) \otimes functor. $\bar{Z}_0: \text{Bord}_{n-1} \rightarrow \text{Fam}_n(C)$ lifting $Z(B_0, \xi_0)$.
- \rightarrow (ii) For each $y \in \text{BO}(n)$, a nondegenerate n -morphism $\eta_y: \mathbb{1} \rightarrow \bar{Z}_0(S^{\xi_y})$ in $\text{Fam}_n(C)$
 lifting. $\eta_y: \mathbb{1} \rightarrow Z(S^{\xi_y})$ in Fam_n .

ξ is universal bundle on $\text{BO}(n)$.

(c) "adjunction" (*) \Rightarrow (i) is equivalent to:

- (i') symmetric monoidal functor. $\text{Bord}_{n-1}(B_0, \xi_0) \rightarrow C$.

Observe (ii) almost looks like the extra data needed to extend a functor $\text{Bord}_{n-1}(B_0, \xi_0) \rightarrow C$

(d) Let $B'_y := \{(x, y, \alpha) \mid x \in B, y \in \text{BO}(n) \text{ and } \xi_x \cong \xi_y \text{ isometry}\}$.

$$\begin{array}{ccc} (x, y, \alpha) & & (x, y, \alpha) \\ \downarrow \text{id} & & \downarrow \text{id} \\ \text{BO}(n) & & B \end{array}$$

For $y \in \text{BO}(n)$, note that

$$B'_y \cong B(B, \xi) \times D\xi_y \cong Z(B, \xi)(D\xi_y)$$

classifying space for (B, ξ) -structures on $D\xi_y$.

For $y \in BO(n)$, note that $B'_y \cong B_{(B, \xi)} D^{2n} \cong Z_{(B, \xi)}(D^{2n}) \cong \eta_y$. (8.5)

Note that (ii) \leftrightarrow For each $y \in BO(n)$, a nondegenerate n -morphism $\eta_y: \mathbb{1} \rightarrow Z_0(S^{2n})$ in $\text{Fam}_n \mathcal{C}$ (lifting $\eta_y: \mathbb{1} \rightarrow Z(S^{2n})$).
unpack. \rightarrow i.e. use "unstraightening"

For all $y \in BO(n)$ & all $y' \in B'_y$, a nondegenerate n -morphism $\eta_{y'}: \mathbb{1} \rightarrow Z_0(S^{2n})$.

(ii') For all $y' \in B'$, a nondegenerate morphism.

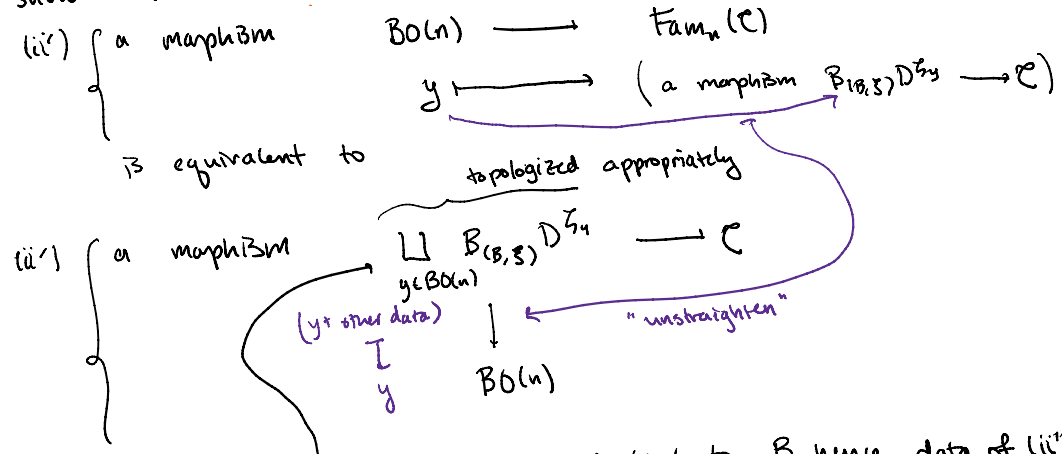
(e). Since for all $x \in B$, $\text{fib}_x(B' \rightarrow B) \cong \{(y, \alpha) : y \in BO(n), \alpha: S_y \cong \xi_x\} \cong EO(n) \cong *$.

$\therefore B' \xrightarrow{\sim} B$ is htpy equivalence.

\Rightarrow (ii'') is equivalent to for all $b \in B$, a nondegenerate n -morphism $\eta_b: \mathbb{1} \rightarrow Z_0(S^{2n})$.

Rmk Steps (d) & (e) are another application of the unstraightening ("adjunction" idea):

(d) show that:



(e) the total space is equivalent to data of (ii''). \rightarrow is homotopy equivalent to B , hence data of (ii').