The Cobordism Hypothesis

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Abstract

A talk at the MIT Juvitop seminar.

1 The Cobordism Hypothesis

$$\mathrm{Fun}^{\otimes}(\mathrm{Bord}_n^{\mathrm{fr}}, \mathcal{C}) \xrightarrow{\sim} (\mathcal{C}^{n\mathrm{d}})^{\sim} := \text{the maximal } \infty\text{-groupoid on the } n\text{-dualizable objects of } \mathcal{C}.$$
$$F \mapsto F(\mathrm{pt})$$

2 Refreshments

Categories will be $(\infty, \text{something})$, often (∞, n). Monoidal categories will be symmetric. An object X in a monoidal category $\mathcal{C}$ is k-dualizable if there exisits

1. (level 0) $\mathbf{1} \xrightarrow{\mathrm{coev}} X \otimes X^\vee$, $X^\vee \otimes X \xrightarrow{\mathrm{ev}} \mathbf{1}$ satisfying Zorro,
2. (level 1) $id_{\mathbf{1}} \xrightarrow{\epsilon} \mathrm{coev}^T \circ \mathrm{coev}$, $\mathrm{coev} \circ \mathrm{coev}^T \xrightarrow{\eta} id_{X \otimes X^\vee}$, same for ev, satisfying Zorro,
3. (level 2) etc, etc
4. ...
5. (level $k-1$) etc, etc.

Lemma 2.0.1. *("Overdualizability") IF $k > n$ then k-dualizability is the same as invertibility.*

Proof. The level n condition tells us that we have adjunctions with units and counits that are equivalences (i.e. isomorphisms in the homotopy category), i.e. the level $(n-1)$ adjunctions were actually equivalences themselves. BUt those are teh units and counits of the level $(n-2)$ adjunctions... □

2.1 A consequence of "overdualizability"

Let's consider maps out of

$$\mathrm{Free}_n^{nd}(\mathrm{pt}) := \text{the free sym. mon. } (\infty, n)\text{-category on one } n\text{-dualizable object.}$$

into a category $\mathcal{C}$. I.e. let's think about

$$\mathrm{Fun}^{\otimes}(\mathrm{Free}_n^{nd}(\mathrm{pt}), \mathcal{C}).$$

Symmetric monoidality implies that $\mathrm{pt} \mapsto$ an n-dualizable object of $\mathcal{C}$. Given two functors F and G, a morphism $\eta : F \to G$ is a symmetric monoidal functor

$$\eta : \mathrm{Free}_n^{nd}(\mathrm{pt}) \to \mathrm{Path}(\mathcal{C}).$$

Since $\mathrm{Path}(\mathcal{C})$ is one categorical dimension lower, if $\mathcal{C}$ was (∞, n) then η lands in the n-dualizable objects of an $(\infty, n-1)$-category. So η is invertible!

Warning 2.1.1. The symmetric monoidal structure on $\mathrm{Path}(\mathcal{C})$ that's being used above, and the one in which η is invertible is the monoidal structure coming from *tensoring* 1-morphisms in $\mathcal{C}$. It is not the same as the usual monoidal structure coming from *composing* 1-morphisms in $\mathcal{C}$.

Luckily, Zorro's equation implies that invertibility for the ⊗-monoidal structure implies invertibility for the composition monoidal structure.

Example 2.1.2. (level 1) $X \in \mathrm{Free}_n^{\mathrm{nd}}(\mathrm{pt})$. If η_X stands for the component of a map $\eta : F \to G$, then $\eta_{X^\vee}$ is its ⊗-inverse and $(\eta_{X^\vee})^\vee$ is its composition inverse:

$$\overset{id}{\frown}\qquad\begin{array}{ccccc} F(X) & \xrightarrow{F(\epsilon\otimes\mathbf{1}_X} & F(X\otimes X^\vee\otimes X) & \xrightarrow{F(\mathbf{1}_X\otimes\eta)} & F(X) \\ & & \downarrow{\scriptstyle\sim} & & \\ & & F(X)\otimes F(X^\vee)\otimes F(X) & & \downarrow{\scriptstyle id} \\ & & \downarrow{\scriptstyle \eta_X\otimes\eta_{X^\vee}\otimes id} & & \\ & & G(X)\otimes G(X^\vee)\otimes F(X) & \xrightarrow{G(\mathbf{1}_X\otimes\eta)} & G(\mathbf{1})\otimes F(X)\simeq F(X) \end{array}.$$

Upshot: If $\mathcal{C}$ is (∞, n) then by "overdualizability, $\mathrm{Fun}^{\otimes}(\mathrm{Free}_n^{nd}(\mathrm{pt}), \mathcal{C})$ is an ∞-groupoid! Moreover, it only sees the n-dualizable objects of $\mathcal{C}$, and by freeness it's exactly that:

$$\mathrm{Fun}^{\otimes}(\mathrm{Free}_n^{nd}(\mathrm{pt}), \mathcal{C}) \simeq (\mathcal{C}^{nd})^{\sim}.$$

So we can reformulate the cobordism hypothesis from above as follows.

2.2 The Cobordism Hypothesis, Framed

The Cobordism Hypothesis, Framed: There is an equivalence of symmetric monoidal (∞, n)-categories $\mathrm{Bord}_n^{\mathrm{fr}} \simeq \mathrm{Free}_n^{nd}(\mathrm{pt})$.

3 Symmetries

Geometrically, $\mathrm{Bord}_n^{\mathrm{fr}}$ carries an $O(n)$ action by rotation of the framing. So the Cobordism Hypothesis implies that $O(n)$ acts on $(\mathcal{C}^{nd})^{\sim}$ for any sym. monoidal category!

3.1 Comparison with stable homotopy

If instead of $(\mathcal{C}^{\mathrm{nd}})^{\sim}$ we take "$GL_1\mathcal{C} := \mathcal{C}^{\times} = (\mathcal{C}^{\infty\mathrm{d}})^{\sim}$, then that is a Picard ∞-groupoid, which is the same as an infinite loop space or "connective spectrum." Such an object X is acted on by the monoid $\Omega^\infty S^\infty$: X comes with a sequence of "deloopings" $X_0 = X, X_1, X_2, ..., X_i \simeq \Omega X_{i+1}$, $X \simeq \Omega^n X_n$, so $\Omega^n S^n$ acts by precomposition and the colimit acts on the colimit.

There is a "J-homomorphism" $O(n) \to \Omega^{n+1} S^{n+1}$ sending an orthogonal matrix to its action on the one-point compactification of $\mathbb{R}^n$, and in the situation that $(\mathcal{C}^{nd})^{\sim} \simeq \mathcal{C}^{\times} = (\mathcal{C}^{\infty\mathrm{d}})^{\sim}$ the Cobordism Hypothesis $O(n)$ action is the same as this one.

Remark 3.1.1. So in a sense, the space of framed n-dimensional fully extended TFTs with fixed target $\mathcal{C}$ is a kind of unstable/truncated version of an infinite loop space or "connective spectrum." Another comment on that at the end...

3.2 Changing the symmetry group

We are motivated in two ways.

1. Geometrically/physically we have a lot of relevant/interesting less structured bordism categories

$$\mathrm{Bord}_n^{(X,\xi)} := \{\text{manifolds } M^d \text{ with a map } M^d \xrightarrow{f} X \text{ and an isomorphism } TM \oplus \mathbb{R}^{n-d} \simeq f^*\xi, \text{ bordisms of those}\}.$$

2. For any topological space Y, not just $Y = \mathrm{pt}$, we can consider the category $\mathrm{Free}_n^{nd}(Y)$, with

$$\mathrm{Fun}^{\otimes}(\mathrm{Free}_n^{nd}(\mathrm{pt}), \mathcal{C}) \simeq \mathrm{Hom}_{\mathrm{Spaces}}(Y, (\mathcal{C}^{nd})^{\sim})$$

   Combining that with the $O(n)$ action and a love for equivariant homotopy theory leads you to conder $O(n)$-spaces Y and equivariant maps.

It turns out you get the best answer you could hope for.

3.3 Cobordism Hypothesis, arbitrarily structured

The Cobordism Hypothesis, Framed: Let $O(n) \hookrightarrow \mathrm{Fr}(\xi) \to X$ be the frame bundle of $\xi \to X$. Note that the fiber of the pullback of ξ is canonically trivial over every point $p \in \mathrm{Fr}(\xi))$ so that each point defined an object in the (X, ξ)-structured bordism category. There is an equivalence of symmetric monoidal $(\infty, 0)$-categories

$$\mathrm{Fun}^{\otimes}(\mathrm{Bord}_n^{(X,\xi)}, \mathcal{C}) \to \mathrm{Hom}_{\mathrm{Spaces}}(\mathrm{Fr}(\xi), (\mathcal{C}^{nd})^{\sim})^{O(n)}$$

$$F \mapsto \big(\mathrm{Fr}(\xi) \ni p \mapsto F(\mathrm{pt} \to p \in \mathrm{Fr}(\xi))\big).$$

3.4 Important special cases

1. $(X, \xi) = (BG, \xi_\rho)$ for a group G and a continuous representation $G \xrightarrow{\rho} O(n)$ with ξ_ρ the associated vector bundle.
2. Contained in the special case above are the even more special cases $G = 1$, $G = O(n)$ with $\rho = id$, and $G = SO(n)$ with $\rho =$ inclusion, which recover framed, unoriented, and oriented bordism.

In that case we just write Bord_n^G and hope that the representation ρ is understood, and the Cobordism Hypothesis says

$$\mathrm{Fun}^{\otimes}(\mathrm{Bord}_n^G, \mathcal{C}) \to \mathrm{Hom}_{\mathrm{Spaces}}(EG \times_G O(n), (\mathcal{C}^{\mathrm{nd}})^{\sim})^{O(n)} \simeq \mathrm{Hom}_{\mathrm{Spaces}}(EG, (\mathcal{C}^{\mathrm{nd}})^{\sim}.)^G \simeq \big((\mathcal{C}^{nd})^{\sim}\big)^{hG}.$$

The last thing is called the "homotopy fixed points."

3.5 From the Cobordism Hypothesis toward GMTW

Suppose that $\mathcal{C}$ is a Picard ∞-groupoid. Then all maps $\mathrm{Bord}_n^G \to \mathcal{C}$ factor through the Picard ∞-groupoid quotient $\mathrm{Bord}_n^G \twoheadrightarrow |\mathrm{Bord}_n^G|$. So the Cobordism Hypothesis says that

$$\mathrm{Hom}_{\Omega^\infty}(|\mathrm{Bord}_n^G|, \mathcal{C}) \simeq \mathrm{Fun}^{\otimes}(\mathrm{Bord}_n^G, \mathcal{C}) \xrightarrow{\sim} \mathcal{C}^{hG}.$$

In other words,$|\mathrm{Bord}_n^G|$ is the infinite loop space corepresenting the functor "take the homotopy G-fixe points." That thing has a name: The infinite loop homotopy quotient of $\Omega^\infty S^\infty$. How do we understand that thing? Well first,

$$\mathcal{C}^{hG} \simeq \mathrm{Hom}_{\mathrm{Spaces}}(EG, \mathcal{C})^G \simeq \mathrm{Hom}_{\Omega^\infty}(\Omega^\infty\Sigma^\infty(EG), \mathcal{C})^G.$$

Now $\Omega^\infty\Sigma^\infty(EG)$ has *two* actions of G: one from before and a new one from the "J-homomorphism" $G \to O(n) \to \Omega^\infty S^\infty$. But those two G actions become indentified on $\mathcal{C}$, so we find ourselves faces with

$$\mathrm{Hom}_{\Omega^\infty}(\text{coequalizer of the } G \text{ actions}, \mathcal{C})$$

Writing $\Omega^\infty\Sigma^\infty(EG)$ as $\Omega^\infty S^\infty \wedge EG$ we get

$$\mathrm{Hom}_{\Omega^\infty}(\Omega^\infty S^\infty \wedge_G EG, \mathcal{C}).$$

Now $\Omega^\infty S^\infty \wedge_G EG$ is the stable spherical fibration associated to a certain map $BG \xrightarrow{\phi} B\Omega_1^\infty S^\infty =$ "$BGL_1\Omega^\infty S^\infty$." Homotopy theorists call that spherical fibration the "Thom spectrum" associated to the map ϕ. Finally, if you stare at it long enough I clam you'll find that

$$\phi = -\rho = BG \xrightarrow{B\rho} BO(n) \hookrightarrow BO \xrightarrow{-1} BO \xrightarrow{J} B\Omega_1^\infty S^\infty.$$

All in all, the Cobordism Hypothesis gives an isomorphism of infinite loop spaces $|\mathrm{Bord}_n^G| \simeq \mathrm{Thom}(-\rho \to BG)$.

3.6 A final comment

We saw above how $\mathrm{Fun}^{\otimes}(\mathrm{Bord}_n^{\mathrm{fr}}, \mathcal{C}) \xrightarrow{\sim} (\mathcal{C}^{n\mathrm{d}})^{\sim}$ is a kind of unstable/truncated version of an infinite loop space, in the sense that $O(n)$ acts as it does through $\Omega^{\infty} S^{\infty}$, if the latter does act. A natural question to ask is: Is there anything more than $O(n)$ that acts on all $(\mathcal{C}^{n\mathrm{d}})^{\sim}$? The answer is yes, $PL(n)$ does. And, when $n \neq 4$, $PL(n)$ is actually it.

That has a weird consequence: $(\mathcal{C}^{n\mathrm{d}})^{\sim}$ differs from $C^{\times}$ only by the existence of dualizable-but-not-necessarily-invertible objects. Just that tiny difference collapses the symmetries that act from $\Omega^{\infty} S^{\infty}$ to $PL(n)$, and the ∞-groupoid $(\mathcal{C}^{n\mathrm{d}})^{\sim}$ knows something about n!