n-fold Complete Segal Spaces

1. Complete Segal Spaces

Recall: Fully faithful nerve functor

$$\begin{aligned} N: \mathrm{Cat} &\longrightarrow \mathrm{SSet} \\ C &\longmapsto \big(n \longmapsto \mathrm{Cat}(\underbrace{\bullet \to \bullet \to \cdots \to \bullet}_{n\times}, C)\big) \end{aligned}$$

$$\vdots$$

$$\downarrow\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow$$

NC_2 = composable pairs of morphisms

$$\downarrow\uparrow\downarrow\uparrow\downarrow$$

NC_1 = morphisms

$$\downarrow\uparrow\downarrow$$

NC_0 = objects

Observe: $\forall\, c, c' \in C_0$:

$$\begin{tikzcd} C(c,c') \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & C_1 \arrow[d] \\ \{(c,c')\} \arrow[r] & C_0 \times C_0 \end{tikzcd}$$

Essential image of $N\colon \mathrm{Cat} \longrightarrow \mathrm{sSet}$: Simplicial sets X s.t.

$\forall\, m,n \in \mathbb{N}_{\geq 0}$:
$$\begin{tikzcd} X_{m+n} \arrow[r] \arrow[d] & X_n \arrow[d] \\ X_m \arrow[r] & X_0 \end{tikzcd}$$
is a pullback.

$$\begin{tikzcd} x_0 \to x_1 \to \cdots \to x_m \to x_{m+1} \to \cdots \to x_{m+n} \arrow[r, mapsto] \arrow[d, mapsto] & x_m \to \cdots \to x_{m+n} \arrow[d, mapsto] \\ x_0 \to x_1 \to \cdots \to x_m \arrow[r, mapsto] & x_m \end{tikzcd}$$

$\boxed{\text{SEGAL CONDITION}}$

An ∞-categegory are a generalisation of categories from set theory to homotopy theory

sets $\rightsquigarrow$ homotopy types a.k.a. <u>spaces</u> (vulg.)
identities / isomorphisms $\rightsquigarrow$ connecting paths / homotopy equivalences

<u>Blackbox</u>: The model category of topological spaces presents the homotopy theory of spaces.

Recall:

A commut. square
$$\begin{tikzcd} W \arrow[r] \arrow[d] & Y \arrow[d, "q"] \\ X \arrow[r, "p"'] & Z \end{tikzcd}$$
of continuous maps of topological spaces is a homotopy pullback if the canonical map

$$W \longrightarrow X \times_Z Z^{\Delta^1} \times_Z Y = \{ (x, \gamma, y) \mid px \overset{\gamma}{\rightsquigarrow} py \}$$

is a weak equivalence.

Replacing "set" with "topological space" and "pullback" with "homotopy pullback" we obtain

DEFINITION: A simplicial topological space $X : \Delta^{op} \longrightarrow Top$ is a Segal space if

$\forall\, m, n \in \mathbb{N}_{>0}$:
$$\begin{tikzcd} X_{m+n} \arrow[r] \arrow[d] & X_n \arrow[d] \\ X_m \arrow[r] & X_0 \end{tikzcd}$$
is a homotopy pullback.

Given a Segal space X, then $\forall\, x, x' \in X_0$ we think of

$$X^h(x,x') := \{(\sigma, f, \sigma') \mid x \overset{\sigma}{\rightsquigarrow} d_1 f,\ d_0 f \overset{\sigma'}{\rightsquigarrow} x'\}$$

as the hom space of x, x'.

We can construct a homotopy category: Obj: Underlying set of X_0.
Mor: $\pi_0 X^h(x,x')$ $\forall\, x, x' \in X_0$

Example: Any homotopically constant simplicial topological space is a Segal space.

In this way we may identify the homotopy theory of spaces as a subtheory of Segal spaces:

$$\mathrm{Top} \xrightarrow{\text{const.}} \mathrm{SegCat} \subseteq \underline{\mathrm{Hom}}(\Delta^{op}, \mathrm{Top})$$

Example: Any category is a Segal space.

<u>New phenomenon:</u> X Segal space, $x, x' \in X_0$ may be equivalent in two different ways:

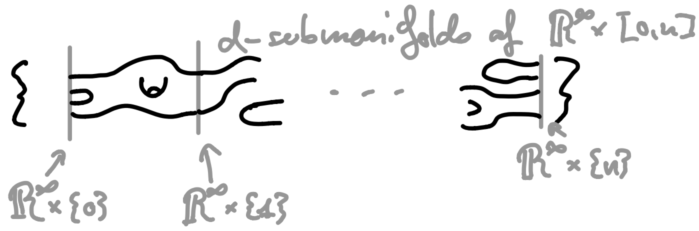
- x, x' are isomorphic: $\exists (\sigma, f, \sigma') \in X^h(x,x')$ which becomes isomorphism in homotopy category.
- x, x' in same path component of X_0.

$$X_0^{\Delta^1} \simeq \mathrm{Path}(x,x') \longrightarrow X^h(x,x') \qquad (*)$$

$$\gamma \longmapsto \left(\gamma|_{[0,\frac{1}{2}]},\ s_0\gamma(\tfrac{1}{2}),\ \gamma|_{[\frac{1}{2},1]}\right)$$

Cartoon:

Segal space Cob_d: $\quad (\mathrm{Cob}_d)_0 = \{$ close $(d-1)$-submanifolds of $\mathbb{R}^\infty\}$

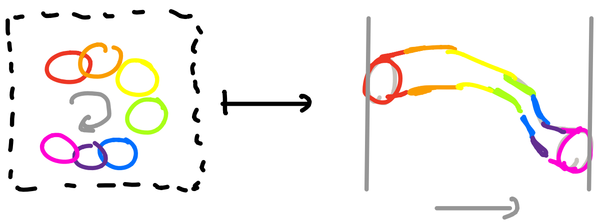
$(\mathrm{Cob}_d)_n = \{$  $\}$

$\triangle_{2}$ no degeneracy maps!

$\forall\, M, N \in (\mathrm{Cob})_0$: $\mathrm{Path}(M,N) \longrightarrow \mathrm{Cob}_d^h(M,N)$ equivalent to:

$\mathrm{Path}(M,N) \longrightarrow (\mathrm{Cob}_d)|_{(M,N)}$

Obs. $(\mathrm{Cob}_d)_1 \to (\mathrm{Cob}_d)_0 \times (\mathrm{Cob}_d)_0$ fibration



DEFINITION: A Segal space X is complete if $\forall\, x, x' \in X_0$:

$\mathrm{Path}(x,x') \longrightarrow X^h(x,x')^{\mathrm{inv.}}$

is a weak equivalence.

NOTATION: $\forall n \geq 1$, $1 \leq k \leq n$, $\alpha_k : [1] \longrightarrow [n]$, $0 \mapsto k-1$, $1 \mapsto k$.
Let X be a Segal space, then $X^{inv} \subseteq X$ subspace consisting of $c \in X_n$: $\alpha_k c$ inv. $\forall n \geq 1$, $1 \leq k \leq n$.

PROPOSITION:

(1) Let X be a Segal space, then
$\forall x, x' \in X_0 : \forall (g, f, g') \in X^h(x, x') : \ (g, f, g')$ inv. $\Leftrightarrow (\mathrm{const}_{d_1 f}, f, \mathrm{const}_{d_0 f})$ inv.

(2) X complete iff $X^{inv.}$ homotopy constant.

Motivation:

Classical: (Rezk) $\underline{\mathrm{Hom}}(\Delta^{op}, \mathrm{Top})$ admits a model structure where fibrant objects are Reed fibrant Segal spaces. Forcing fully faithful essentially surjective functors to be w.e produces new model structure where fibrant objects are the complete Segal spaces.

Modern: $N(\bullet \rightleftarrows \bullet) \longrightarrow *$ should be an equivalence.

But the bordism example shows that we don't necessarily want to complete. See Ayala-Francis for modern perspective.

2. Category objects

DEFINITION: A <u>category object</u> in a category C is a simplicial diagram $\Delta^{op} \longrightarrow C$ satisfying the Segal condition.

EXAMPLE:

1. A category object in SET is a category.
2. A category object in Top is a topological category ⚠
3. A category object in CAT is a double category.

Observe that SET is (canonically) a full subcategory of both Top and CAT.

$\rightsquigarrow$
- A topologically enriched category X is the same as a topological category X s.t. $X_0 \in \mathrm{SET}$.
- A 2-category X is the same as a double category s.t. $X_0 \in \mathrm{SET}$.

3. n-fold Complete Segal Spaces

3.1 Inductive definition

<u>$n=0$</u>: A 0-fold complete Segal space is a topological space.

<u>$n=1$</u>: A 1-fold complete Segal space is a complete Segal space.

(Analogy: A Segal object in SET is a category)

$\underline{n=2}$: Recall: $\mathrm{Top} \xhookrightarrow{\text{const.}} \mathrm{CSS} \subseteq \underline{\mathrm{Hom}}(\Delta^{op}, \mathrm{Top})$

A 2-fold complete Segal space is a simplicial object

$$X: \Delta^{op} \longrightarrow \mathrm{CSS}$$

such that:

a) X satisfies Segal condition (homotopy pullback computed levelwise)

b) X_{0*} homotopically constant.

c) X_{*0} complete

Idea: Extract underlying Segal space of X:

$$n \longmapsto X_{n*}^{inv}$$

$\underline{n \geq 2}$: And so on.

3.2 Not so inductive definition

$\underline{n=0}$: A 0-fold complete Segal space is a topological space.

$\underline{n=1}$: A 1-fold complete Segal space is a complete Segal space.

$\underline{n\geq 2}$ A n-fold complete Segal space is a functor $\Delta^{op} \times \cdots \times \Delta^{op} \longrightarrow Top$

a) $\forall\, 1 \leq i \leq n \;\; \forall\, k_1, \dots, k_{i-1}, k_{i+1}, \dots, k_n \geq 0$

$$X_{k_1, \dots, k_{i-1}, \bullet, k_{i+1}, \dots, k_n}$$

is a Segal space.

b) $\forall\, 1 \leq i \leq n \; \forall\, k_1, \dots, k_{i-1} \geq 0$

$$X_{k_1, \dots, k_{i-1}, 0, \bullet, \dots, \bullet}$$

is homotopically constant.

c) $\forall\, 1 \leq i \leq n \; \forall\, k_1, \dots, k_{i-1} \geq 0$

$$X_{k_1, \dots, k_{i-1}, \bullet, 0, \dots, 0}$$

is a complete Segal space.