TCFT & String Topology

Recall Folklore Thm

- "2D-TFT <=> Frobenius algebras"

  A : fin. dim. commutative k-alg.
  with $tr: A \to k$ such that
  the pairing $A \otimes A \xrightarrow{m} A \xrightarrow{tr} k$
  is nondegenerate

- $\mathrm{Fun}^{\otimes}(\mathrm{Cob}(2), \mathrm{Vect}_k) \simeq \mathrm{Frob}(k)$

Fully - extended version

$$\mathrm{Fun}^{\otimes}(\mathrm{Bord}_2, \mathcal{C}) \simeq \mathcal{C}^{\sim}$$

$\uparrow$ ∞-groupoid of fully dualizable objects

Today : Topological conformal field theories

$Z: \mathrm{Bord}_2^{nc} \to \mathcal{C}$ <~> Calabi-Yau object $Z(*)$ of $\mathcal{C}$.

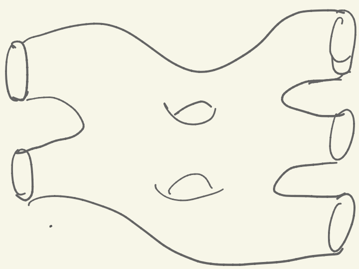
- <u>Segal PROP</u> $\mathcal{M}$

  - <u>Objects</u> : finite sets

  - <u>Morphisms</u> : $Mor_{\mathcal{M}}(I, J) = \coprod_{\Sigma} \mathcal{M}_{\Sigma}$

    - Σ is a surface with $|I|$ incoming, $|J|$ outgoing circle boundaries s.t. each connected component of Σ has nonempty incoming boundary.

    - $\mathcal{M}_{\Sigma}$ is the moduli space of conformal structures over Σ.



$\in Mor_{\mathcal{M}}(\underline{2}, \underline{3})$.

not a morphism.

- <u>Def</u>. A <u>CFT</u> is a sym. mon. functor

  $Z : \mathcal{M} \to Vect_{\mathbb{C}}$.

• **Linearize** $\Rightarrow$ $C_*(\mathcal{M})$. same objects

$$\mathrm{Mor}_{C_*(\mathcal{M})}(I, J) = C_*\Big(\coprod_{\Sigma} \mathcal{M}_\Sigma\Big)$$
$$\simeq C_*\Big(\coprod_{\Sigma} B\mathrm{Diff}^+_\partial(\Sigma)\Big).$$

Here $\mathrm{Diff}^+_\partial(\Sigma) = \{$ orientation-preserving diffeos that equal the identity on (a collar nbhd) of the boundary$\} /$ isotopies.

⚠ Conformal str. irrelevant!

Def A **(closed) TCFT** is a sym. mon. functor

$$C_*(\mathcal{M}) \longrightarrow Ch_k$$

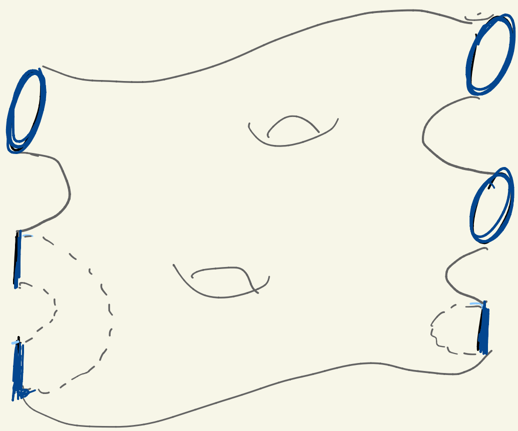
Def An **HQFT** - - - - -

$$H_*(\mathcal{M}) \longrightarrow Gr\,Mod_k$$

cf. String topology w/ string operations

Upgrade => Open-closed TCFT

- $\mathcal{OC}$:
  - Objects: oriented 1-mflds w/ boundary
  - Morphisms: $C_*\left(\bigsqcup_{\Sigma} \mathcal{M}_\Sigma\right)$



s.t. each connected component has nonempty incoming boundary.

- Let $\mathcal{O} \subset \mathcal{OC}$ be the full subcat. with objects finite intervals.

Thm of Costello

$k = \mathbb{Q}[x, x^{-1}]$, $|x| = d$.

1). $\{$Open TCFTs $Z : \mathcal{O} \to Ch_k\}$

$\simeq \{$Calabi-Yau A_∞-categories A on one object$\}$

Rmk. This is the categorification of Frobenius algebras. Informally, a CY A_∞-cat on c is a stable $(\infty,1)$-cat with $tr : C(c,c) \to k$ that induces a nondegenerate symmetric pairing $\langle f, g \rangle = tr(g \circ f)$.

2). Given an open TCFT Z_0, there is a universal open-closed TCFT $Z : \mathcal{OC} \to S$ via left Kan extension. s.t. $Z(S^1)$ is the Hochschild complex of A.

- Fully extended version

$(\infty, 2)$

$$\{\text{Sym. mon. } Z : \mathrm{Bord}_2^{nc} \to \mathcal{C}\}$$

$$\Longleftrightarrow \{\text{Calabi-Yau objects of } \mathcal{C}\}$$

$$Z \mapsto Z(*)$$

- Bord_2^{nc} (← non-compact) is, informally, the $(\infty, 2)$-cat
w/ - objects : oriented 0-mflds X
  - 1-mor : oriented bordisms $B : X \to X'$
  - 2-mor : oriented bordisms $\Sigma : B \to B'$
    s.t. Σ is trivial along X, X'
    and every connected component of Σ
    has nonempty intersection w/ B.
  - Higher mor invertible.

• **Def.** A **Calabi-Yau** object of $\mathcal{C}$ consists of

• a dualizable object $X \in \mathcal{C}$ with

(exercise)

• an $SO(2)$-equivariant morphism

$\eta : \dim(X) = ev_X \circ coev_X \to 1$ in $\Omega\mathcal{C}$

that is the counit for an adjunction

$ev_X \dashv coev_X$.

e.g. For $\mathcal{S}$ a "good" $(\infty,1)$-cat. (e.g. Ch_k)

$Alg_{(1)}(\mathcal{S})$ is the $(\infty,2)$-cat w/ objects

the E_1-algebras of $\mathcal{S}$.

• A **CY object** of $\mathcal{S}$ is then

an E_1-alg A with $SO(2)$-equivariant

$tr : \int_{S^1} A = \dim(A) \longrightarrow 1$ s.t.

$$A \otimes A \simeq \int_{S^0} A \to \int_{S^1} A \xrightarrow{tr} 1$$

identifies A with $A^\vee = A^{op}$.

Rmk. A fully dualizable object $Y \in \mathcal{C}$

$\iff$ a dualizable object Y s.t. ev_Y has both left & right adjoint.

- For a CY object X, ev_X only needs a right adjoint corresponding to the 2-morphism , while  is not a 2-morphism.

- On the other hand, CY objects of $\mathcal{C}^{\wedge}$ are the $SO(2)$-fixed pts.

Example : String Topology

Main player : $\mathcal{L}M = M^{S^1}$, where M^{2d} is oriented and simply-connected.

Take 1. $C^*(M)$ is an E_1-alg in Mod_k. $k = \mathbb{Q}[x, x^{-1}]$. $|x| = d$

$$tr : \int_{S^1} C^*(M) \simeq C^*(\mathcal{L}M) \xrightarrow{i^*} C^*(M) \xrightarrow{[M]} k$$

$\rightsquigarrow$ $(C^*(M), tr)$ is CY and determines a TFT $\mathrm{Bord}_2^{nc} \to \mathrm{Alg}_{(1)}(\mathrm{Mod}_k)$.

Q. Does this recover the classical construction of string operations?

[Godin]: $H_*(\mathcal{L}M)$ is an HCFT.

i.e. there are compatible maps

$$H_*(\mathcal{M}_\Sigma(I, J)) \otimes H_*(\mathcal{L}M)^{|I|} \to H_*(\mathcal{L}M)^{|J|}$$

parametrizing string operations.

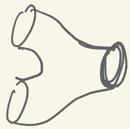
e.g. Chas-Sullivan loop product

$$H_*(\mathcal{L}M) \otimes H_*(\mathcal{L}M) \to H_*(\mathcal{L}M)$$

is defined as follows:

- Consider the correspondence diagram

$$\mathcal{L}M \xleftarrow{\rho_{out}} \mathrm{Map}(P, M) \xrightarrow{\rho_{in}} \mathcal{L}M \times \mathcal{L}M$$

where P is the bordism  .

We want to construct an umkehr map $(\rho_{in})_!$.

- [Cohen-Jones]

Take $X = \mathcal{L}M \times_M \mathcal{L}M \simeq \mathrm{Map}(P, M)$, which sits in

$$\begin{tikzcd} X \arrow[r, hook, "e"] \arrow[d, "ev"'] & \mathcal{L}M \times \mathcal{L}M \arrow[d, "ev \times ev"] \\ M \arrow[r, hook, "\Delta"] & M \times M \end{tikzcd}$$

, a pullback of fiber bundles.

Let V be a tubular nbhd of Δ, which pulls back to the total space of the normal bundle $\eta_e : ev^*(V) \to X$. Thom collapse yields $H_*(\mathcal{L}M \times \mathcal{L}M) \to H_*(Th(\eta_e)) \to H_*(X)$.

<u>Take 2</u>. $C_*(\Omega M)$ is an E_1-alg in Mod_k

w/ $\int_{S^1} C_*(\Omega M) \simeq C_*(\mathcal{L}M)$.

However, $C_*(\Omega M)$ is not often dualizable.

e.g. $H_*(\Omega S^2) \cong k[x]$.

- One can fix this by taking the opposite target category $\mathrm{Alg}_1^{op}(\mathrm{Mod}_k)$, and there is a cotrace $k \to C_*(\mathcal{L}M)$ given by the fundamental class $[M]$.

- Thus we get a TFT

$$Z: \mathrm{Bord}_2^{nc} \to \mathrm{Alg}_1^{op}(\mathrm{Mod}_k).$$

⚠ Unclear if this recovers string operations, or how to check it.

<u>AFAIK</u>, Blumberg-Cohen-Teleman come the closest to constructing a lift.