Factorization algebra as an extended TFT

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This are notes for a talk given during the Juvitop seminar in Fall 2020. The main references are

- Lurie's paper "On the classification of Topological Field Theories"
- Scheimbauer's Ph.D. thesis "Factorization Homology as a Fully Extended Topological Field Theory"

Throughout, we let $(\mathcal{C}, \otimes)$ be a symmetric monoidal ∞-cat; we assume that $\mathcal{C}$ admits sifted colimits and that $- \otimes -$ preserves them in each component. We fix a natural number $n \in \mathbb{N}$.

1 E_n-algebras

We start by recalling the definition of an E_n-algebra. Let $\mathrm{Man}_n^{\mathrm{fr}}$ the symmetric monoidal ∞-category obtained as the coherent nerve of the topological category with

- objects: framed n-dimensional smooth manifolds;
- morphism spaces $\mathrm{Emb}^{\mathrm{fr}}(M, N)$ of framed embeddings $M \hookrightarrow N$ between manifolds.
- The monoidal product is given by disjoint union.

We denote by $\mathrm{Disk}_n^{\mathrm{fr}} \subset \mathrm{Man}_n^{\mathrm{fr}}$ the full ∞-subcategory spanned by those framed n-manifolds which are isomorphic to a finite disjoint union $\coprod_{\mathrm{finite}} \mathbb{R}^n$ (with the standard framing).

Definition 1. *An* E_n***-algebra*** *in* $\mathcal{C}$ *is a symmetric monoidal functor from* $\mathrm{Disk}_n^{\mathrm{fr}}$ *to* $\mathcal{C}$*.*

If $A\colon \mathrm{Disk}_n^{\mathrm{fr}} \to \mathcal{C}$ is an E_n-algebra, we say that $A(\mathbb{R}^n) \in \mathcal{C}$ is the underlying object of A; by abuse of notation, we also denote it by A. Unraveling the definition, we see that A is equipped with multiplication maps

$$A^{\otimes k} = A \otimes \cdots \otimes A \simeq A(\mathbb{R}^n \sqcup \cdots \sqcup \mathbb{R}^n) \longrightarrow A(\mathbb{R}^n) = A \tag{1}$$

parameterized by the space of framed embeddings $\mathrm{Emb}^{\mathrm{fr}}(\mathbb{R}^n \sqcup \cdots \sqcup \mathbb{R}^n, \mathbb{R}^n)$ of k disks into a bigger disk.

For example, for $n = 1$, the space $\mathrm{Emb}^{\mathrm{fr}}(\coprod^k \mathbb{R}^1, \mathbb{R}^1)$ is homotopy equivalent to the discrete set of permutations of $\{1, \ldots, k\}$; hence the multiplication maps are parameterized by choosing a linear order on the input copies of A. Unraveling the coherence conditions, one sees that an E_1-algebra in $\mathcal{C}$ is precisely an associative (and unital) algebra object.

The goal of this talk is to explain a construction which takes as input an E_n-algebra A and produces an n-dimensional topological field theory $\mathrm{Bord}_n^{\mathrm{fr}} \to \mathrm{Alg}_n$ with values in an (∞, n)-category Alg_n which we can heuristically describe as follows:

- The objects of Alg_n are E_n-algebras;
- a 1-morphism $A \to A'$ between E_n-algebras is an A-A'-bimodule, i.e., an $\mathrm{E}_{(n-1)}$-algebra B on which A and A' act compatibly from the left and from the right, respectively;
- a 2-morphism

$$\begin{array}{c} A \\ {}^{B}\big\downarrow \;\overset{C}{\Rightarrow}\; \big\downarrow^{B'} \\ A' \end{array} \tag{2}$$

  between two bimodules $B, B' \colon A \to A'$ is a B-B'-bimodule C, by which we mean an $\mathrm{E}_{(n-2)}$-algebra C on which A, A', B, B' act compatibly from the top, the bottom, the left and the right, respectively;
- 3-morphisms are bimodules of bimodules of bimodules of E_n-algebras;
- ...
- n-morphisms are (bimodules of)^n of E_n-algebras.
- Composition in Alg_n is defined as a suitable tensor product which, for instance, sends an A-A'-bimodule B and an A'-A''-bimodule B' to the A-A''-bimodule $B \otimes_{A'} B'$.

The topological field theory associated to the E_n-algebra A is supposed to send the point $\mathbb{R}^0 \in \mathrm{Bord}_n^{\mathrm{fr}}$ to the object $A \in \mathrm{Alg}_n$. According to the cobordism hypothesis, this property uniquely characterizes this TFT. We will in fact give an explicit formula to compute this TFT on an arbitrary k-morphism M in $\mathrm{Bord}_n^{\mathrm{fr}}$ $(0 \leq k \leq n)$ as the *factorization homology* $\int_{M \times \mathbb{R}^{n-k}} A$; heuristically, we take the local data encoded by the E_n-algebra A and "integrate" it over the n-manifold $M \times \mathbb{R}^{n-k}$.

2 Factorization Algebras

The first challenge is to give a rigorous definition of the (∞, n)-category Alg_n. One approach makes use of *factorization algebras*, which we introduce now.

Let X be a topological space. Denote by $\mathcal{U}_X$ the colored operad[1] with

- colors/objects are the open subsets of X;
- there is a unique operation/morphism $U_1, \ldots, U_k \to U$, whenever $U_1, \ldots, U_k \subseteq \mathcal{U}_X$ are *pairwise disjoint* subsets of $U \in \mathcal{U}_X$.

Definition 2. *A* ***prefactorization algebra*** *F on X with values in $\mathcal{C}$ is an $\mathcal{U}_X$-algebra in $\mathcal{C}$, i.e. a map of ∞-operads $\mathcal{U}_X \to \mathcal{C}$[2]. Unraveling the definition, F assigns an object $F(U) \in \mathcal{C}$ to each open subsets $U \in UX$, and a morphism $F(U_1) \otimes \cdots \otimes F(U_k) \to F(U)$, whenever $U_1, \ldots, U_k$ are pairwise disjoint open subsets of $U \in \mathcal{U}_X$; it needs to be functorial in the obvious sense.*

A ***factorization algebra*** *is a prefactorization algebra F which additionally satisfies*

1. *If $U_1, \ldots, U_k \in \mathcal{U}_X$ are pairwise disjoint open subsets of X, the induced map $F(U_1) \otimes \cdots \otimes F(U_k) \xrightarrow{\simeq} F(U_1 \sqcup \cdots \sqcup U_k)$ is an equivalence (in particular, for $k = 0$, the object $F(\emptyset)$ is identified with the monoidal unit of $\mathcal{C}$).*
2. *A suitable descent condition, which allows the value $F(U)$ to be computed as a colimit of values on sufficiently well behaved open covers. We shall not spell it out here.*

A factorization algebra F on X is called ***locally constant****, if the inclusion $F(D) \to F(D')$ is an equivalence, whenever $D \subseteq D'$ are both (homeomorphic to) $\mathbb{R}^n$.*

3 Factorization homology

The following construction shows how an E_n-algebra gives rise to a factorization algebra on each framed n-manifold M.

Construction 1. *Let $A\colon \mathrm{Disk}_n^{\mathrm{fr}} \to \mathcal{C}$ be an E_n-algebra. We denote by*

$$\left(\int_{-} A\right) : \mathrm{Man}_n^{\mathrm{fr}} \to \mathcal{C} \tag{3}$$

[1] One way to think of a colored operad is a "multi-category" which has objects (usually called colors) and between them not just 1-to-1-morphisms $x \to y$, but also many-to-1-morhphisms $(x_1, \ldots, x_k) \to y$. It needs to satisfy the suitable analogs of associativity and unitality.

[2] Each symmetric monoidal $(\infty\text{-})$category is canonically an $(\infty\text{-})$operad, by declaring a multi-morphism $x_1, \ldots x_k \to y$ to simply be a morphism $x_1 \otimes \cdots \otimes x_k \to y$ in $\mathcal{C}$.

the left Kan extension of A and call it ***factorization homology with coefficients in*** *A. For any framed n-manifold M, it is computed explicitly by the pointwise formula:*

$$\int_M A := \operatorname{colim}\left(\mathrm{Disk}_n^{\mathrm{fr}}/M \to \mathrm{Disk}_n^{\mathrm{fr}} \xrightarrow{A} \mathcal{C}\right) \tag{4}$$

where $\mathrm{Disk}_n^{\mathrm{fr}}/M$ denotes the overcategory of all possible embeddings of disjoint disks into M. By construction, $\int_M A$ is functorial along embeddings of manifolds; hence in particular along inclusions of open subsets. Moreover one can check that the ∞-category $\mathrm{Disk}_n^{\mathrm{fr}}/M$ is sifted, hence the monoidal product in $\mathcal{C}$ commutes with the limit in (4)*; a direct calculation produces a canonical identification*

$$\left(\int_{U_1} A\right) \otimes \cdots \otimes \left(\int_{U_k} A\right) \xrightarrow{\simeq} \left(\int_U A\right) \tag{5}$$

whenever $U = U_1 \sqcup \cdots \sqcup U_k$ arises as a pairwise disjoint union of open subsets $U_1, \ldots, U_k$ of M. This exhibits $\int_{-\subseteq M} A$ as a factorization algebra on M. It is locally constant because the inclusion $D \subseteq D'$ of two disks is an equivalence in the ∞-category $\mathrm{Man}_n^{\mathrm{fr}}$.

An important special case arises when we consider $M = \mathbb{R}^n$. In this case, we have $\int_{\mathbb{R}}^n A = A$ and in fact the factorization algebra $\int_{-\subseteq\mathbb{R}^n} A$ on $\mathbb{R}^n$ encodes the same data as the E_n-alegebra A. More precisely we have the following theorem.

Theorem 1 (Lurie)**.** *The assignment $A \mapsto \int_{\_} A$ assembles to an equivalence of ∞-categories between E_n-algebras in $\mathcal{C}$ and locally constant factorization algebras on $\mathbb{R}^n$ with values in $\mathcal{C}$.*

4 Stratified factorization algebras

To systematically encode the (bimodules of ...) which make up the (∞, n)-category Alg_n, it is convenient to study a statified variant of factorization algebras.

Let X be a topological space. A stratification of X consists of an ascending chain $\emptyset = X_{-1} \subset X_0 \subset X_1 \subset \cdots \subset X_l = X$ of closed subspaces. The **index** of an open subset $U \subset X$ is the smallest i such that $U \cap X_i \neq \emptyset$.

Definition 3. *Let X be a topological space with stratification $X_\bullet$. A factorization algebra F on X is called* ***locally constant with respect to the stratification****, if the inclusion $D \subseteq D'$ induces an equivalence $F(D) \xrightarrow{\simeq} F(D')$ whenever D and D' are disks of the same index i which both remain connected when intersected with X_i.*

Note that we get the previous notion of locally constancy with respect to the trivial stratification $\emptyset \subset X$.

Finally, let us remark that factorization algebras which are locally constant with respect to stratifications can be pushed forward along suitable maps

$f\colon X \to Y$ of stratified spaces by declaring $f_* F(U) = F(f^{-1}(U))$ for each open subset $U \subset Y$.

5 The Morita category

We can now finally say, at the very least, what the morphisms are in the (∞, n)-category Alg_n.

For each k, a k-morphism in Alg_n is a factorization algebra on $\mathbb{R}^n$ which is locally constant with respect to the stratification

$$S^k\colon \emptyset \subset \cdots \subset \emptyset \subset \mathbb{R}^{n-k} \times \{0\}^k \subset \cdots \subset \mathbb{R}^{n-1} \times \{0\} \subset \mathbb{R}^n. \tag{6}$$

Inside the stratified space (6) we find the two subspaces

$$\mathbb{R}^{n-k} \times (-\infty, 0) \times \mathbb{R}^{k-1} \subset \mathbb{R}^n \quad \text{and} \quad \mathbb{R}^{n-k} \times (0, +\infty) \times \mathbb{R}^{k-1} \subset \mathbb{R}^n \tag{7}$$

which are both isomorphic as stratified spaces to $(\mathbb{R}^n, S^{k-1})$. Thus we can restrict each factorization algebra F on $(\mathbb{R}^n, S^k)$ to two factorization algebras on $(\mathbb{R}^n, S^{k-1})$ which we declare to be the source and target $(k-1)$-morphism of F, respectively.

The composition in Alg_n can be roughly described as follows: Given two composable k-morphisms $E \xrightarrow{F} E' \xrightarrow{F'} E''$, we can reparameterize them and glue them to a factorization algebra on the stratified space

$$\emptyset \subset \cdots \subset \emptyset \subset \mathbb{R}^{n-k} \times \{-1, 1\} \times \{0\}^{k-1} \subset \mathbb{R}^{n-k+1} \times \{0\}^{k-1} \subset \cdots \subset \mathbb{R}^{n-1} \times \{0\} \subset \mathbb{R}^n, \tag{8}$$

where E, E', E'' are identified with the restriction to

$$\mathbb{R}^{n-k} \times (-\infty, -1) \times \mathbb{R}^{k-1}, \tag{9}$$

$$\mathbb{R}^{n-k} \times (-1, 1) \times \mathbb{R}^{k-1}, \tag{10}$$

$$\mathbb{R}^{n-k} \times (+1, +\infty) \times \mathbb{R}^{k-1}, \tag{11}$$

respectively. This factorization algebra can then be pushed forward along the map $\mathbb{R}^n \to \mathbb{R}^n$ which in the $(n-k+1)$-th coordinate sends $[-1, 1]$ to 0 and identifies

$$+1\colon (-\infty, -1] \xrightarrow{\cong} (-\infty, 0] \quad \text{and} \quad -1\colon [+1, +\infty) \xrightarrow{\cong} [0, +\infty). \tag{12}$$

6 Factorization homology as a TFT

Finally we sketch how to make $\int_{\_} A$ into a functor of (∞, n)-categories

$$\left(\int_{\_} A \right) : \mathrm{Bord}_n^{\mathrm{fr}} \to \mathrm{Alg}_n. \tag{13}$$

If we are given a k-morphism N in $\mathrm{Bord}_n^{\mathrm{fr}}$, we can consider the factorization algebra $\int_{-\subseteq M} A$ on $M := N \times \mathbb{R}^{n-k}$. For $k = 0$, i.e., $N = \mathbb{R}^0$ gives rise to the factorization algebra $\int_{-\subseteq \mathbb{R}^n} A$ which is exactly the object corresponding to A in Alg_n.

For $k \neq 0$, we have to push forward along a suitable map to a stratified space by choosing appropriate collars. For example, if we are given a 1-morphism, i.e. a cobordism N between N_0 and N_1, we can choose collars

$$N_0 \times (-\infty, 0] \hookrightarrow N \hookleftarrow N_1 \times [0, +\infty) \tag{14}$$

and define a map $f\colon N \to \mathbb{R}$ as follows:

- on the collars it is given by projecting onto $(-\infty, 0]$ or $[0, +\infty)$, respectively;
- all other points go to 0.

Finally, we can define the value of our TFT on N to be the factorization algebra obtained by pushing $\int_{-\subseteq N \times \mathbb{R}^{n-1}}$ forward along $f \times \mathrm{id}\colon N \times BR^{n-1} \to \mathbb{R}^n$.

The construction for higher k is similar by repeatedly choosing collars in $M := N \times \mathbb{R}^{n-k}$ and then pushing forward along an analogous collapse map $M \to \mathbb{R}^n$, where the right side is stratified as in (6). See the following picture for $k = n = 2$:

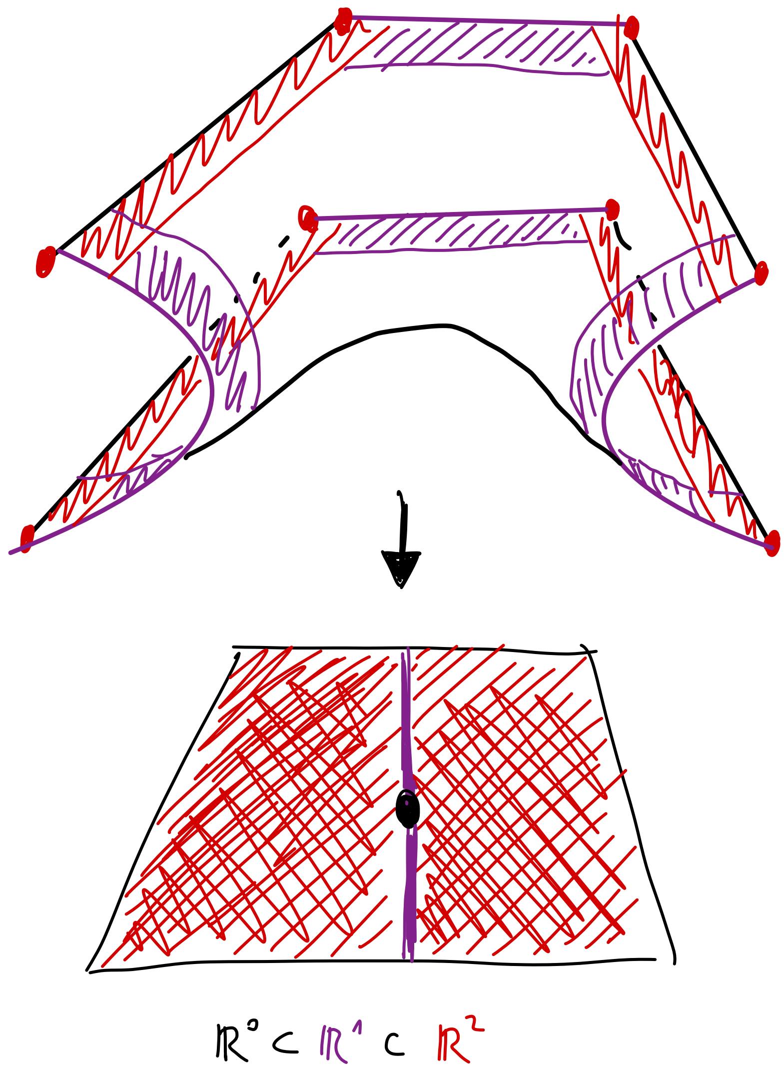

Figure 1: An example of a 2-morphism in $\mathrm{Bord}_2^{\mathrm{fr}}$ with collars and the associated collapse map to the stratified space $\mathbb{R}^2$