# COBORDISM HYPOTHESIS INTRODUCTION

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Abstract. This talk is based on pages 2-15 of Lurie [3]. We introduce Atiyah's definition of a topological field theory and examine what data a TFT provides in dimensions 1 and 2. Using these examples, we motivate Baez and Dolan's Cobordism Hypothesis [1].

Contents



The goal of this talk is to formulate Baez and Dolan's Cobordism Hypothesis [1] and discuss why you might believe it. Lurie's introduction in [3] is wonderfully readable and most of this talk is mainly just a reformatting of Lurie's intro to fit into a juvitop talk.

## 1. Modeling Field Theories

Whatever the Cobordism Hypothesis says, it has something to do with field theories. We start by figuring out what a field theory should be. A good overview can be found in the introduction of Costello-Gwilliam's book, [2, Ch.1 §4]. There are many different mathematical theories for modeling field theories. Each model has to account for a few basic features. For this, it's good to keep in mind the picture of a particle moving around in a box.

- Space: What world we're in, what area your system is varying over
- Time: how long you are letting the system run

When you combine the manifold describing space (say $M$) with the manifold describing time (say $\mathbb{R}$) you get a new manifold $M \times \mathbb{R}$ that is cleverally called "spacetime."

- Fields: the possible paths a particle could take
  For $U \subset X \times \mathbb{R}$, we get $\text{Fields}(U) = \text{Maps}(U, \text{Box})$.
- Rules: what makes the system special? What constraints are there on the paths particles can take? For example (in the massless free field theory), this might say that the particle has to follow a straight line. In general, these cut out a subspace of fields that satisfy the equation.
  These rules are called the *equations of motion* or the *Euler-Lagrange equations*. It is the differential equation determined by the *action functional*.
  ex. (free massless theory) the particle must move in a straight line

Solutions to these equations define a subspace

$$\mathrm{Sol}(U) \subset \mathrm{Fields}(U)$$

- Observables: measurements you can take on the solutions to the Euler-Lagrange equations.

$$\mathrm{Obs}(U) = \mathrm{Hom}(\mathrm{Sol}(U), \mathbb{R})$$

- Statistical numbers: In a *topological* field theory, these numbers don't depend on the metric of spacetime.

  These are called *correlation functions.* The main one of importance for us will be the *partition function.*

| **Features** | **Classical Mechanics** | **Atiyah/Segal** |
|---|---|---|
| Space | a fixed manifold $X$. In quantum mechanics, the Hilbert space of states | any oriented $(n-1)$-dimensional manifold, including the emptyset |
| ? | ? | Value of $Z$ on any oriented $(n-1)$-dimensional manifold |
| Time | $\mathbb{R}$ or an interval | the closed interval $[0, 1]$ |
| Spacetime | a fixed $n$-dimensional manifold $Y$ Think $Y = X \times \mathbb{R}$ | any oriented $n$-dimensional manifold with boundary. |
| Fields | For $U \subset X \times \mathbb{R}$ open, $\mathrm{Maps}(U, \mathrm{Box})$ | ? |
| Equations of Motion | differential equation determining subspace $\mathrm{Sol}(U) \subset \mathrm{Fields}(U)$. | ? |
| Observables | $C^\infty(\mathrm{Sol}(U), \mathbb{R})$ | $Z(S^{n-1})$. |
| Partition Function | ? | value of $Z$ on a closed $n$-manifold |

I've left a few boxes blank. We can talk about these in discussion section next week.

*Notation.* For $X$ an oriented manifold, let $\overline{X}$ denote $X$ with the opposite orientation.

**Definition 1.1.** Let $n$ be a positive integer. We define a category $\mathbf{Cob}(n)$ as follows:

- objects: $(n-1)$-dimensional oriented manifolds
- morphisms from $M$ to $N$ are given by equivalence classes of $n$-dimensional oriented manifolds with boundary $B$ together with an an orientation-preserving diffeomorphism

$$\partial B \simeq \overline{M} \sqcup N$$

  Two morphisms $B$ and $B'$ are equivalent if there is an orientation-preserving diffeomorphism $B \to B'$ that restricts to the identity on the boundaries,

$$\begin{array}{ccc} B & \longrightarrow & B' \\ \downarrow \simeq & & \downarrow \simeq \\ \overline{M} \sqcup N & = & \overline{M} \sqcup N \end{array}$$

Composition is given by gluing cobordisms along their shared boundary. View $\mathbf{Cob}(n)$ as a symmetric monoidal category under disjoint union.

For $k$ a field, let $\mathbf{Vect}(k)$ denote the symmetric monoidal category of $k$-vector spaces with tensor product.

**Definition 1.2** (Atiyah, Segal)**.** Let $k$ be a field. A *topological field theory* (TFT) of dimension $n$ is a symmetric monoidal functor $Z\colon \mathbf{Cob}(n) \to \mathbf{Vect}(k)$.

In particular, $Z(\emptyset) = k$.

*Remark* 1.3. Note that Atiyah's definition considers all possible spacetimes at once, instead of working on one specific $n$-manifold at a time. We could instead consider the category of bordism submanifolds within a fixed manifold.

What makes these field theories *topological* is the lack of a Hamiltonian. We only care about the cobordism $n$-manifolds as smooth manifolds, not Riemannian manifolds.

*Notation.* For $V$ a $k$-vector space, let $V^\vee$ denote the linear dual, $V^\vee := \mathrm{Hom}(V, k)$.

Let $Z$ be an $n$-dimensional TFT. Given an oriented $(n-1)$-manifold $M$, the product manifold $M \times [0,1]$ can be viewed as a morphism in $\mathbf{Cob}(n)$ in multiple ways.

- As a morphism from $M \to M$, the product $M \times [0,1]$ maps to the identity map
$$\mathrm{id}\colon Z(M) \to Z(M).$$
- As a morphism $M \sqcup \overline{M} \to \emptyset$, the product $M \times [0,1]$ determines an evaluation map
$$\mathrm{ev}\colon Z(M) \otimes Z(\overline{M}) \to k.$$
- As a morphism $\emptyset \to \overline{M} \sqcup M$, the product $M \times [0,1]$ determines a coevaluation map
$$\mathrm{coev}\colon k \to Z(\overline{M}) \otimes Z(M).$$

Recall that a pairing $V \otimes W \to k$ is perfect if it induces an isomorphism $V \to W^\vee$.

**Proposition 1.4.** *Let $Z$ be a topological field theory of dimension $n$. For every $(n-1)$-manifold $M$, the vector space $Z(M)$ is finite dimensional. The evaluation map $Z(M) \otimes Z(\overline{M}) \to k$, induced from the cobordism $M \times [0,1]$, is a perfect pairing.*

Proving this is one of the problems we'll talk about in next weeks discussion section.

## 2. Classifying Topological Field Theories

### 2.1. **Low Dimensions.**

**Example 2.1** (Dimension 1)**.** Let $Z\colon \mathbf{Cob}(1) \to \mathbf{Vect}(k)$ be a 1-dimensional TFT. Let $P$ denote a single point with positive orientation and $Q = \overline{P}$. Let $Z(P) = V$. This finite-dimensional vector space, determines $Z$ on objects. By Proposition 1.4, $Z(Q) = Z(\overline{P}) = V^\vee$. A general object of $\mathbf{Cob}(1)$ looks like
$$M = \coprod_{S_+} P \sqcup \coprod_{S_-} Q$$
for $S_+, S_-$ sets. Since $Z$ is symmetric monoidal, we have
$$Z(M) = \coprod_{S_+} V \sqcup \coprod_{S_-} V^\vee$$
What about morphisms? A morphism in $\mathbf{Cob}(1)$ is a 1-dimensional manifold with boundary $B$. Using the monoidal structure, it suffices to describe $Z(B)$ where $B$ is connected. There are five possibilities

- $B$ is an interval viewed as a morphism $P \to P$. Then $Z(B) = \mathrm{Id}_V$.

- $B$ is an interval viewed as a morphism $Q \to Q$. Then $Z(B) = \mathrm{Id}_{V^\vee}$.
- $B$ is an interval viewed as a morphism $P \sqcup Q \to \emptyset$. Then $Z(B)\colon V \otimes V^\vee \to k$. By Proposition 1.4, this is the canonical pairing of $V$ and $V^\vee$,

$$Z(B)(x \otimes f) = f(x)$$

  Under the isomorphism $V \otimes V^\vee \cong \mathrm{End}(V)$, the morphism $Z(B)$ corresponds to taking the trace.
- $B$ is an interval viewed as a morphism $\emptyset \to P \sqcup Q$. Then $Z(B)$ is the map

$$k \to V \otimes V^\vee \cong \mathrm{End}(V)$$

  sending $\lambda \in k$ to $\lambda \mathrm{Id}_V$.
- $B = S^1$ is a circle viewed as a morphism $\emptyset \to \emptyset$. Then $Z(S^1)$ is a linear map $k \to k$; i.e., multiplication by some $\gamma \in k$. To determine $\gamma$, view $S^1$ as the union of two semi-circles along $P \sqcup Q$. This determines a decomposition of $S^1$ into the composite of two cobordisms,

$$\emptyset \to P \sqcup Q \to \emptyset$$

  By the above cases, $Z$ maps this to the composite

$$k \to \mathrm{End}(V) \to k$$

  of the map $\lambda \mapsto \lambda \mathrm{Id}_V$ and the trace map. Thus $Z(S^1)$ is the scaling by $\mathrm{Tr}(\mathrm{Id}_V) = \dim(V)$ map.

*Remark* 2.2. In 1-dimension, the observables are given by $Z(S^0) = \mathrm{End}(V)$. Notice that this has the structure of an associative algebra.

Thus in dimension 1, we see that the vector space $Z(P)$ determines the TFT $Z$. Does every $V \in \mathbf{Vect}(k)$ appear as $Z(P)$ for some 1-dimensional TFT? Nope, only the finite-dimensional ones. We get an equivalence of categories

$$\mathrm{Fun}^{\otimes}(\mathbf{Cob}(1), \mathbf{Vect}(k)) \to \mathbf{Vect}^{\mathrm{fin}}(k)$$

by evaluating on the point.

Let's try to do something similar in dimension 2.

**Example 2.3** (Dimension 2)**.** Let $Z$ be a 2-dimensional TFT. The only objects in $\mathbf{Cob}(2)$ are the empty set and disjoint unions of copies of $S^1$. We don't get a new object $\overline{S^1}$ since the circle has an orientation-reversing diffeomorphism. The observables, $A = Z(S^1)$ determines $Z$ on objects.

What about morphisms? A morphism in $\mathbf{Cob}(2)$ is 2-dimensional oriented manifold with boundary.

- The pair of pants cobordism determines a map $m\colon A \otimes A \to k$. One can check that $m$ defines a commutative, associative multiplication on $A$.
- The disk $\mathbb{D}^2$ viewed as a cobordism $S^1 \to \emptyset$ determines a linear map $\mathrm{Tr}\colon A \to k$.
- The disk $\mathbb{D}^2$ viewed as a cobordism $\emptyset \to S^1$ determines a linear map $k \to A$. The image of $1 \in k$ under this map acts as a unit for the multiplication. Indeed, we can glue $\mathbb{D}^2$ to one of the legs of the pants. The resulting manifold is diffeomorphic to $S^1 \times [0,1]$. But $S^1 \times [0,1]$ maps to $\mathrm{Id}_A$ under $Z$.

Note that the composite of

$$A \otimes A \xrightarrow{m} A \xrightarrow{\mathrm{Tr}} k$$

comes from the cobordism $S^1 \times [0,1]$ viewed as a map $S^1 \sqcup S^1 \to \emptyset$. By Proposition 1.4, the map $\mathrm{Tr} \circ m$ is a nondegenerate pairing.

**Definition 2.4.** A *commutative Frobenius algebra* over $k$ is a finite-dimensional commutative $k$-algebra $A$, together with a linear map $\mathrm{Tr}\colon A \to k$ such that the bilinear form $(a, b) \mapsto \mathrm{Tr}(a, b)$ is nondegenerate.

**Theorem 2.5.** *The category of 2-dimensional TFTs is equivalent to the category of Frobenius algebras.*

*Remark* 2.6. In 2-dimensions, the observables $Z(S^1)$ of a TFT $Z$ has the structure of a Frobenius algebra.

2.2. **Higher Dimensions.** The problem when we try to classify TFTs in higher dimensions is that the objects become too complicated. Up to reversing orientation and taking disjoint unions, the categories $\mathbf{Cob}(1)$ and $\mathbf{Cob}(2)$ have a unique object, $P$ and $S^1$, respectively. For $n = 3$, there are infinitely many oriented 2-manifolds, one for each genus $g$. We don't think of genus $g$ surfaces as being that complicated. In fact, we usually think of $\Sigma_g$, the genus $g$ surface, as coming from $g$ connect sums of the torus. A closely related way to say this, is that $\Sigma_g$ has a relatively easy handle-body decomposition. But what happens when we view $\Sigma_g$ under its handle-body decomposition? We're really viewing it as a composition of cobordisms; i.e., as a morphisms in $\mathbf{Cob}(2)$. Similarly, when we tried to understand the value of a 2-dimensional field theory on $S^1$, we broke $S^1$ into the union of two semi-circles, that is to say, into its handle-body decomposition.

If we want to be understand a $n$-dimensional field theory by breaking manifolds down, using their handle-body decompositions, into lower-dimensional manifolds, we need the TFT to know about manifolds of dimension $< n - 1$. In particular, we would like some sort of data assigned to every $(n - 2)$-dimensional manifold and we would like this data to have something to do with the values on $(n - 1)$-manifolds.

The way to encode all this data is the language of higher categories.

**Definition 2.7.** A *strict $n$-category* is a category $\mathcal{C}$ enriched over $(n - 1)$-categories.

For $n = 2$, this means that for objects $A, B \in \mathcal{C}$ the morphisms $\mathsf{Hom}_{\mathcal{C}}(A, B)$ is itself a category.

**Example 2.8.** The strict 2-category $\mathbf{Vect}_2(k)$ has objects cocomplete $k$-linear categories and morphisms

$$\mathrm{Hom}_{\mathbf{Vect}_2(k)}(C, D) = \mathbf{Fun}_k^{\mathrm{cocon}}(C, D)$$

the functor category of cocontinuos, $k$-linear, functors.

**Example 2.9.** The strict 2-category $\mathbf{Cob}_2(n)$ has

- objects: closed, oriented manifolds of dimension $n - 2$.
- morphisms: $\mathrm{Hom}_{\mathbf{Cob}_2(n)}(X, Y) =: \mathcal{C}$ should be the category with
  - objects: cobordisms $X \to Y$
  - morphisms: $\mathrm{Hom}_{\mathcal{C}}(B, B')$ is equivalence classes of bordisms $X$ from $B \to B'$

The big problem here is making the composition law strictly associative. One would like to define composition by gluing bordisms, but get messed up in defining a smooth structure on the result, and things that used to be equalities are now just homeomorphisms. The solution will be to get rid of the "strictness" and move to $(\infty, 2)$-categories.

**Definition 2.10.** Let $\mathcal{C}$ be a symmetric monoidal $n$-category. An *extended* $\mathcal{C}$-valued $n$-dimensional TFT is a symmetric monoidal functor

$$Z\colon \mathsf{Cob}_n(n) \to \mathcal{C}$$

The purpose of this definition is to allow us to reduced $n$-dimensional TFTs down to information about 1-dimensional TFTs. As we saw before, a 1-dimensional TFT is determined by its value on a point. Thus we might make the following guess.

**Guess.** *An extended field theory is determined by its value on a single point. Moreover, evaluation on a point determines an equivalence of categories between TFTs valued in $\mathcal{C}$ and $\mathcal{C}$.*

There's two problems with this guess.

(1) Even in 1-dimension, not every vector space determined a TFT. We needed to restrict to finite-dimensional ones. The analogue in higher dimensions will be something called "fully dualizable objects."
(2) Orientation in dimension 1 is the same as a framing. This isn't true in higher dimensions. We actually wanted a framing, not just an orientation so that we could say that locally $M^k$ was canonically diffeomorphic to $\mathbb{R}^k$ (via the exponential map). Thus we need a version of $\mathsf{Cob}_n(n)$ that works with framed manifolds instead of oriented ones.

**Theorem 2.11** (Baez-Dolan Cobordism Hypothesis: Framed Version)**.** *Let $\mathcal{C}$ be a symmetric monoidal $(\infty, n)$-category with duals. Then the evaluation functor $Z \mapsto Z(*)$ induces an equivalence*

$$\mathsf{Fun}^{\otimes}(\mathbf{Bord}_n^{\mathrm{fr}}, \mathcal{C}) \to \mathcal{C}^{\sim}$$

*between framed extended $n$-dimensional TFTs valued in $\mathcal{C}$ and the fully dualizable subcategory of $\mathcal{C}$.*

## References


[1] John C. Baez and James Dolan. Higher-dimensional algebra and topological quantum field theory. *J. Math. Phys.*, 36(11):6073–6105, 1995.

[2] Kevin Costello and Owen Gwilliam. *Factorization algebras in quantum field theory. Vol. 1*, volume 31 of *New Mathematical Monographs*. Cambridge University Press, Cambridge, 2017.

[3] Jacob Lurie. On the classification of topological field theories. In *Current developments in mathematics, 2008*, pages 129–280. Int. Press, Somerville, MA, 2009.