GMTW (HOMOTOPY TYPE OF THE COBORDISM CATEGORY) JUVITOP PROBLEMSESSION

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Moduli spaces of manifolds

Exercise 0.1 (Parametrizing tangential structures)**.** For a $\mathrm{GL}_n(\mathbb{R})$-equivariant space Θ, a structure on a smooth fibre bundle $\pi : E \to X$ with n-dimensional fibers is a GL_n-equivariant map $\mathrm{Fr}(T_\pi E) \to \Theta$, where $T_\pi E$ is the vertical tangent bundle $\ker D\pi$.

(a) Show that having a Θ structure as defined above implies we have a lift of the tangent classifier of each fiber $\pi^{-1}(x)$ along a fibration $B \to BGL_d$. (Be very geometric.)
(b) For what GL_n-space Θ would a Θ structure be the data of a smoothly varying family of orientations on the fibers of $\pi : E \to X$
(c) Now find Θ such that a Θ-structure is a smoothly varying family of framings.

Definition 0.2. A *concordance* between two fibre bundles $\pi_0 : E_0 \to X$ and $\pi_1 : E_1 \to X$ with Θ-structures $\rho_0 : \mathrm{Fr}(T_\pi E_0) \to \Theta, \rho_1 : \mathrm{Fr}(T_\pi E_1) \to \Theta$ is a fibre bundle $\pi : E \to X \times \mathbb{R}$ with isomorphisms of $(\pi_0, \rho_0), (\pi_1, \rho_1)$ to the pullbacks along $\{0\} \times X \hookrightarrow \mathbb{R} \times X, \{1\} \times X \hookrightarrow \mathbb{R} \times X$.

Exercise 0.3 (Classifying space of fibre bundles with Θ-structure)**.** Consider the contravariant functor in the category of manifolds taking X to the set $\mathscr{F}[X]$ of concordance classes of fibre bundles over X. Let Δ^k_e be the open standard p-simplex $\{t \in \mathbb{R}^{k+1} | \sum t_i = 1\}$ and consider a simplicial set $F^\Theta_\bullet$ whose p-simplices are the set of smooth fibre bundles $E \to \Delta^p_e$ with Θ-structure. Show that its geometric realization is a classifying space for $\mathscr{F}$, i.e.

$$\mathscr{F}[X] \cong [X, |F^\Theta_\bullet|]$$

What does each connected component of $|F^\Theta_\bullet|$ classify? (Hints [1] [2])

The moduli space $|F^\Theta_\bullet|$ is denoted $\mathcal{M}^\Theta$. In general, for any sheaf $\mathcal{F} : \mathrm{Man}^{\mathrm{op}} \to$ Sets on the category of manifolds, this procedure produces a space classifying its concordance classes. For instance, we can recover the familiar notion of a classifying space of principal G-bundles.

Exercise 0.4 (BG)**.** Show the construction above recovers our usual notion of BG obtained from geometrically realizing the nerve of G, or whatever you know it

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[1]Hint 1: Define a simplicial set $F^\Theta_\bullet(X)$ with p-simplices fibre bundles with Θ-structure over $X \times \Delta^p_e$. Show $\mathscr{F}^\Theta[X] \cong \pi_0 F^\Theta_\bullet(X)$.

[2]Hint 2: Find a natural way to produce maps $\mathrm{Sing}(X) \to F^\Theta_\bullet$.

as. (Note concordance and diffeomorphism are equivalent in the case of principal G-bundles).

Of course with principal G-bundles we can do one better, because BG also comes with a universal fibration $EG \to BG$ such that taking pullbacks of classifying maps gives us back our bundles. There is also such a universal fibration for the moduli space of manifolds above, or any similar construction with pullbacks. The universal bundle $\mathcal{E}^\Theta$ is given by the geometric realization of the simplicial set that has p-simplices the set of triples (π, ρ, s) where $\pi : E \to \Delta^p_e$ is a fibre bundle, ρ is a Θ-structure, and s is a section of π.

Exercise 0.5 (Universal fibre bundle with Θ structure)**.** Show there is a diagram

$$\begin{array}{ccccc} E & \xleftarrow{\simeq} & |\mathrm{Sing}(E)| & \longrightarrow & \mathcal{E}^\Theta \\ \downarrow & & \downarrow & & \downarrow \\ M & \xleftarrow{\simeq} & |\mathrm{Sing}(M)| & \longrightarrow & \mathcal{M}^\Theta \end{array}$$

Exercise 0.6. Read Notation 1.4.3 of Page 19 of Lurie and give a model for the fibre bundle $E \to \mathcal{B}(M, N)$.

The topological cobordism category

Exercise 0.7. Notation 1.4.3 (Exercise 0.6) topologizes bordism sets of Cob(n), but does it topologically enrich the category? Assuming you followed the moduli of manifolds construction from above, there should be one issue. (Hint: [3]) Try to fix it by topologizing the morphism sets in a different way. (Hints: [4] [5])

Exercise 0.8. Now topologize the object set of Cob(n) also.

A category with a topology on objects and morphisms is called a *topological category*. One reason to topologize Cob(n) rather than just topologically enrich it is making the space $B\mathrm{Cob}_{\mathbb{R}^n}(d)$ into an E_n-algebra, and upgrading the equivalence in Madsen-Weiss to a map of E_n-algebras (more on this soon).

Exercise 0.9 (Contains spoilers to previous 2 exercises probably)**.** We defined a topological bordism category $\mathcal{C}_d(n)$ by embedding the objects into $\mathbb{R}^n$ and morphisms into $\mathbb{R}^n$ times some interval. Show $B\mathcal{C}_d(n)$ has an E_n structure. We will usually work with the direct limit over all n, $\mathcal{C}_d$.

GMTW Lore

Two big predecessors to GMTW are in the low dimensions: the Madsen-Weiss theorem (homological stability of mapping class groups) and, before that, Barratt-Priddy-Quillen (homological stability of symmetric groups).

[3] composition

[4] In problem 2 we saw each connected component of the moduli space of manifolds classifies fibre bundles with fibre W for some diffeomorphism class of n-manifolds W. Then $\mathcal{M}^\Theta = \bigsqcup_W \mathrm{BDiff}(W)$. Find a way to build $\mathrm{BDiff}(W)$ that is more compatible with composition.

[5] Embed things. Is that enough?

Lurie sort of mentions MW but let's go over the result in a little more detail: Let C_g be the space of subsurfaces of $(-\infty, g] \times \mathbb{R}^\infty$ diffeomorphic to the genus g surface with one boundary component $\Sigma_{g,1}$, with some prescribed boundary circle. (Note taking the space of subsurfaces is quite different from taking the space of embeddings)

Exercise 0.10. C_g is a $K(\pi_0 \mathrm{Diff}(\Sigma_{g,1}), 1)$.

We have $C_g \subset C_{g+1}$ by attaching twice punctured tori in $[g, g+1] \times \mathbb{R}^\infty$ with prescribed boundary circles, and $\mathcal{C}_\infty = \cup_g C_g$ the limit, then MW showed

$$H_*(\mathcal{C}_\infty) \cong H_*(\Omega_0^\infty AG_{2,\infty}^+)$$

induced by a map $\alpha : \mathcal{C}_\infty \to \Omega^\infty AG_{2,\infty}^+$ where $AG_{2,\infty}^+$ is the affine grassmanian of affine 2 planes in $\mathbb{R}^\infty$. Unlike BPQ though, the map in MW applies to arbitrary dimension and is related to the map giving the homotopy equivalence in GMTW.

Exercise 0.11 (Scanning map)**.** Let N be a d-manifold and $C(N, \mathbb{R}^n)$ the space of submanifolds of $\mathbb{R}^n$ diffeomorphic to N. Construct a map $C(N, \mathbb{R}^n) \to \Omega^n AG_{d,n}^+$ by noting any embedded $N \subset \mathbb{R}^n$ is kindof planar if you look really really close! Stabilize to a map $C(N, \mathbb{R}^\infty) \to \Omega^\infty AG_{d,\infty}$ which specializes to the MW map in the case $d = 2$.

Ok, so what about GMTW? In the previous exercises we've constructed a topological cobordism category

$$\mathrm{ob}\, \mathcal{C}_d \simeq \coprod_{\substack{\text{diffeo} \\ \text{classes} \\ M}} B\mathrm{Diff}(M) \qquad \mathrm{mor}\, \mathcal{C}_d \simeq \coprod_{\substack{\text{diffeo} \\ \text{classes} \\ W}} B\mathrm{Diff}(W, \partial)$$

as a colimit over categories $\mathrm{Cob}_{\mathbb{R}^n}(d)$ whose objects are embedded d-manifolds in $\mathbb{R}^{n+d-1}$ and morphisms are embedded manifolds with boundary in $[a, b] \times \mathbb{R}^{n+d-1}$. Let $G(d, n)$ be the Grassmanian of d planes in $\mathbb{R}^{n+d}$ and $U_{d,n}^\perp$ the bundle $\{(V, v) \in G(n, d) \times \mathbb{R}^{n+d} | v \perp V\}$. GMTW shows there's an equivalence

$$\alpha : B\mathcal{C}_d \to \Omega^{\infty-1} MTO(d) \tag{1}$$

where $MTO(d)$ is the spectrum whose $(n+d)$th space is $\mathrm{Th}(U_{n,d}^\perp)$. In fact there's equivalences $B\mathrm{Cob}_{\mathbb{R}^n}(d) \to \Omega^{n+d-1}\mathrm{Th}(U_{d,n}^\perp)$. We showed the LHS is an E_{n+d-1} algebra, and the RHS is too. Chris Schommer-Pries showed how to find a zig zag of weak equivalences between these two spaces that are all maps of E_{n+d-1}-algebras.

The map α above goes like this: a morphism in the topological cobordism category is a bordism $W \subset [a_0, a_1] \times \mathbb{R}^{n+d-1}$. Thom collapse map of its normal bundle ν gives a map $[a_0, a_1]_+ \wedge S^{n+d-1} \to \mathrm{Th}(\nu)$ which we can compose with the normal classifier $\mathrm{Th}(\nu) \to \mathrm{Th}(U_{d,n}^\perp)$. Taking the adjoint to get $[a_0, a_1]_+ \wedge S^0 \to \Omega^{n+d-1}\mathrm{Th}(U_{n,d}^\perp) \to \Omega^{\infty-1} MTO(d)$. Then a morphism in $\mathcal{C}_d$ gives us a path in $\Omega^{\infty-1} MTO(d)$. This assembles into a functor $\mathcal{C}_d \to \mathrm{Path}(\Omega^{\infty-1} MTO(d))$ and taking B gives $\alpha : B\mathcal{C}_d \to \Omega^{\infty-1} MTO(d)$.

Exercise 0.12. How are the α from GMTW and the α from MW related?

A bit about the GMTW paper

GMTW proves the equivalence between the spaces in 1 by constructing a zig-zag of equivalences between *sheaf models* for these spaces. What does that mean!

Exercise 0.13 (Possibly very hard even though I've tried being suggestive throughout the handout, pls ask me for hints I guess)**.** The goal is to construct a Cat-valued sheaf $C_d : Man^{\mathrm{op}} \to Cat$ with a continuous functor $|C_d| \to \mathcal{C}_d$ to the topological cobordism category that is a weak equivalence on classifying spaces.

(1) For each manifold X, construct a "topological cobordism category over X" $C_d(X)$ where the objects are fibre bundles over X. Think about what you need to make things glue as effortlessly as possible. (Hints: [6] [7])
(2) Show the functor $C_d : Man^{\mathrm{op}} \to Cat$ is isomorphic to $C^\infty(-, \mathcal{C}_d)$, so we have a continuous functor $|C_d| \to \mathcal{C}_d$.
(3) Show $N_k|C_d| \to N_k\mathcal{C}_d$ is an equivalence.

GMTW also proves a version of the above theorem with Θ-structures. In Problem 1 we defined Θ-structure on a fibre bundle $\pi : E \to X$ and showed (spoilers) it is equivalent to a lift of the classifying map for $T_\pi E$ along a fibration $\theta : B = \Theta//GL_d \to G(d,\infty)$ (which in particular means we lift the tangent classifiers of each fiber). Let $MT^\theta(d)$ be the spectrum with $(n+d)$th space $\mathrm{Th}(\theta^* U^\perp_{d,\infty})$, then

$$\alpha : B\mathcal{C}^\theta_d \to \Omega^{\infty-1} MT^\theta(d)$$

Exercise 0.14. (a) Show that in the case of framed bordism, the spectrum $MT^\theta(d)$ is a shifted sphere spectrum.
(b) Be happy because, as Lurie remarks in page 51, —Bord$_n$— is loops infinity of the sphere, and the bordism category in GMTW is a $d-1$-fold looping of Lurie's extended bordism category.

[6]Before we embedded objects in $a_i \times \mathbb{R}^{n+d-1}$ and morphisms in $[a_0, a_1] \times \mathbb{R}^{n+d-1}$, now we should have an X-parametrixed family of such embeddings.
[7]For things to glue nicely we probably want ϵ-sized collars at first, though then we can take limit as $\epsilon \to 0$