DISCUSSION SECTION 1

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Abstract. Below you'll find a collection of questions. You should think of these questions less as a homework assignment and more as a playground to run around in and spark your imagination. It's more fun playing on the swings with friends, so come to discussion section to talk with others about anything you find confusing or exciting! Each section is self-contained, so feel free to jump around rather than reading linearly.

Contents

1. Playing with **Cob**(n) 1
2. Strict 2-Categories 2
3. Field Theory Models 2
4. Observables 2
5. ∞-Category Basics 3
References 5

1. Playing with **Cob**(n)

Question 1. Check that composition is well-defined in **Cob**(n). What is the smooth structure on the gluing of two cobordisms?

Question 2. Is there a difference between the words cobordism and bordism?

Question 3. Let Z be a topological field theory of dimension n. Show that for every $(n-1)$-manifold M, the vector space $Z(M)$ is finite dimensional. Check that the pairing $Z(M) \otimes Z(\overline{M}) \to k$, induced from the cobordism $M \times [0,1]$, is perfect.

Recall that a pairing $V \otimes W \to k$ is perfect if it induces an isomorphism $V \to W^\vee$. The following is [1, Exercise III.9.0.7].

Question 4. Show that collection of TFTs forms a space, which informally means that every symmetric monoidal natural transformation $Z \to Z'$ is actually a natural equivalence.

Question 5. Prove that the category of 2-dimensional TFTs is equivalent to the category of commutative Frobenius algebras. In particular, show that a TFT Z sends the pair of pants cobordism to a commutative, associative multiplication on $Z(S^1)$.

Question 6. Study the data of a a 2-dimensional TFT on a surface of genus g for $g > 2$ like what is done in [5, Ex. 1.2.1.].

2. Strict 2-Categories

Question 7. Read Definition 1.2.9 of [5]. Show that a strict 1-category is just an ordinary category.

Question 8. What should the strict 2-category $\mathbf{Vect}_2(k)$ analogue of $\mathbf{Vect}(k)$ be?

Question 9. What difficulties do you see arising in making the definition of $\mathbf{Cob}_2(n)$ rigorous? Can we still define composition by gluing? What is the smooth structure? Compare with Question 1.

Question 10. Define a symmetric monoidal structure on $\mathbf{Vect}_2(k)$ and on $\mathbf{Cob}_2(n)$.

For $\mathcal{C}$ a symmetric monoidal strict 2-category, define $\Omega\mathcal{C} = \mathbf{Maps}_{\mathcal{C}}(1, 1)$.

Question 11. What is $\Omega\mathbf{Vect}_2(k)$? What about $\Omega\mathbf{Cob}_2(n)$?

3. Field Theory Models

Question 12. Try filling out the rest of the table comparing features of classical mechanics and Atiyah's model from the talk.

Question 13. What other definitions of TFTs have you heard before? Can you match them up with Atiyah's formulation?

Question 14. What makes Atiyah's TFT's topological?

Question 15. What's the difference between a TFT and a TQFT?

4. Observables

Recall that the *observables* of an n-dimensional TFT Z is given by $Z(S^{n-1})$.
Imagine we had an ∞-category version $\mathcal{C}\mathrm{ob}(n)$ of $\mathbf{Cob}(n)$ and replace $\mathbf{Vect}(k)$ with Ch_k.

Question 16. The observables $Z(S^0) = \mathrm{End}(Z(P))$ of a 1-dimensional TFT Z has the structure of an associative algebra. If we upgrade Z to a map of ∞-categories

$$Z\colon \mathcal{C}\mathrm{ob}(1) \to \mathrm{Ch}_k$$

how does the algebraic structure on $Z(S^0)$ change?

Question 17. In 2-dimensions, the observables $Z'(S^1)$ of a TFT Z' has the structure of a Frobenius algebra. If we upgrade Z to a map of ∞-categories

$$Z\colon \mathcal{C}\mathrm{ob}(1) \to \mathrm{Ch}_k$$

how does the algebraic structure on $Z(S^1)$ change?

Question 18. Can you predict a pattern from the algebraic structure of $Z(S^1)$ and $Z(S^0)$? What algebraic structure should the observables of an n-dimensional field theory have in the ∞-context?

Question 19. Don't do this question until you've tried Question 18. The idea behind the cobordism hypothesis is to build up a n-dimensional TFT inductively from lower-dimensional manifolds and cobordisms. Given the relation you found in Question 18, how do you think this induction relates to Dunn additivity?

See [7, §2.34.1] for a statement of Dunn additivity. See [3, Thm. 5.1.2.3] for a proof and a relation to another Baez-Dolan hypothesis.

5. ∞-Category Basics

If you've never used ∞-categories before, you might want to read pages 15-24 of [5].

Question 20. Why can't we just use topological categories instead of $(\infty, 1)$-categories? For one answer, see Incorrect Definition 1.4.5 of [5] and the reasons it's "incorrect."

5.1. **Modeling ∞-categories as simplicial sets.** Recall that the nerve functor $\mathrm{N}\colon \mathbf{Cat}_1 \to \mathbf{sSet}$ from ordinary categories to simplicial sets is *fully faithful*.

Question 21. Let C be a category. Show that the nerve $\mathrm{N}(C)$ of C satisfies the following *inner horn filling condition*: for all integers $n > 0$ and $0 < k < n$ and all maps $f\colon \Lambda^n_k \to \mathrm{N}(C)$, there exists a dotted lift

$$\begin{array}{ccc} \Lambda^n_k & \xrightarrow{f} & \mathrm{N}(C) \\ \big\downarrow & \overset{\bar{f}}{\nearrow} & \\ \Delta^n & & \end{array} .$$

Is this dotted lift unique? (First think about the cases where $n = 2$ and $n = 3$ and draw some pictures.)

Question 22. Let C be a category. Show that C is a groupoid if and only if the nerve $\mathrm{N}(C)$ is a Kan complex.

Given simplicial sets X and Y, write $\mathrm{Hom}(X, Y)$ for the internal Hom in simplicial sets, defined by

$$\mathrm{Hom}(X, Y) := \mathbf{sSet}(X \times \Delta^\bullet, Y) .$$

Question 23. Let C be a category. Show that the restriction map

$$\mathrm{Hom}(\Delta^2, \mathrm{N}(C)) \to \mathrm{Hom}(\Lambda^2_1, \mathrm{N}(C))$$

is an isomorphism of simplicial sets. (Can you prove a statement before taking nerves that implies this?)

The following result gives two ways of formulating the definition of a *quasicategory* or *∞-category.*

Theorem 5.1 (Joyal [4, Corollary 2.3.2.2])**.** *The following are equivalent for a simplicial set X:*

(1) *For all integers $n > 0$ and $0 < k < n$ and all maps $f\colon \Lambda^n_k \to X$, there exists a dotted lift*

$$\begin{array}{ccc} \Lambda^n_k & \xrightarrow{f} & X \\ \big\downarrow & \overset{\bar{f}}{\nearrow} & \\ \Delta^n & & \end{array} .$$

(2) *The restriction map*

$$\mathrm{Hom}(\Delta^2, C) \to \mathrm{Hom}(\Lambda^2_1, C)$$

is a trivial Kan fibration of simplicial sets.

Question 24. How does condition (2) encode the idea that the composition of morphisms in an ∞-category is defined 'up to a contractible space of choices'?

5.2. **Examples of ∞-categories.**

Question 25 (spans)**.** Let C be a category with pullbacks (if you want to be more concrete, take C to be the category of finite sets). There is a $(2,1)$-category $\mathrm{Span}(C)$ of *spans* of C with:

(0) Objects the objects of C.
(1) A 1-morphism $x_0 \to x_1$ in $\mathrm{Span}(C)$ is a diagram

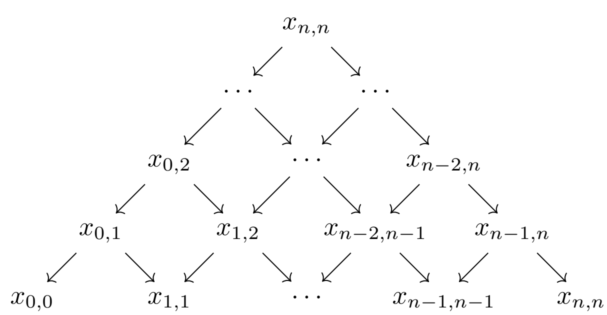
$$\begin{array}{ccc} & y & \\ \swarrow & & \searrow \\ x_0 & & x_1 \end{array}$$

in C. Such a diagram is called a *span*. Composition is (weakly) defined by pullback:

$$\left(\begin{array}{ccc} & z & \\ \swarrow & & \searrow \\ x_1 & & x_2 \end{array}\right) \circ \left(\begin{array}{ccc} & y & \\ \swarrow & & \searrow \\ x_0 & & x_1 \end{array}\right) := \begin{array}{ccccccc} & & & y \times_{x_1} z & & & \\ & & \swarrow & & \searrow & & \\ & y & & & & z & \\ \swarrow & & \searrow & & \swarrow & & \searrow \\ x_0 & & & x_1 & & & x_2 \end{array} .$$

(2) 2-morphisms are isomorphisms of spans.

The point of this exercise, is to work through how to regard $\mathrm{Span}(C)$ as an ∞-category. Thinking about how the nerve of a category works, try to define a simplicial set $\mathrm{Span}(C)_\bullet$ with n-simplices diagrams



in C such that each square is a pullback. What are the face and degeneracy maps? What categories index diagrams of this shape, and what's an easy way to make them into a simplicial object?

Show that the simplicial set you've defined is an ∞-category.

Question 26. Can the definition of $\mathrm{Span}(C)_\bullet$ that you gave in Question 25 be adapted to the case where C is an ∞-category with pullbacks?

Question 27. Do you think the ∞-category $\mathrm{Span}(\mathbf{Fin})_\bullet$ of spans of finite sets has finite products or coproducts? Describe them.

If you're interested take a look at [2, §4.2] for a brief desctiption of how to use $\mathrm{Span}(\mathbf{Fin})_\bullet$ to encode E_∞-structures.

Question 28. If you're not familiar with it, look up the definition of a *differential graded (dg) category* (for example, see [6, Tag 000P9]). Think about how you might try to define a simplicial set that captures all of the homotopical information about a dg category.

After thinking about this for a while, look up the definition of the *dg nerve* in Kerodon [6, Tag 000PK].

5.3. **Straightening/unstraightening.** Let $\mathbf{Cat}_\infty$ denote the ∞-category of ∞-categories and write $\mathbf{Space} \subset \mathbf{Cat}_\infty$ of the ∞-category of spaces (∞-groupoids).

Question 29. Let C be an ∞-category. Why might it be hard to write down a functor $C \to \mathbf{Space}$ or $C \to \mathbf{Cat}_\infty$ 'by hand'? How would you write down a functor $C \to \mathbf{Cat}_\infty$ that on objects is given by $C \mapsto C_{/c}$ and on morphisms is given by sending $f\colon c \to c'$ to the functor $C_{/c} \to C_{/c'}$ given by composition with f.

Question 30. Read through §1 of Dylan Wilson's tutorial on ∞-categories [8]. Try to work through the 1-categorical version of the straightening/unstraightening equivalence.

Question 31. What is the universal left fibration $U \to \mathbf{Space}$? In the case of left fibrations over $[1]$, check that for every functor $X\colon [1] \to \mathbf{Space}$, pullback along the universal left fibration $U \to \mathbf{Space}$ you came up with agrees with the cylinder construction of the morphism $X_0 \to X_1$.

5.4. **Working toward complete Segal spaces.**

Question 32. Let C be a category. Show that for each integer $n > 0$, the map

$$\mathrm{N}(C)[n] \to \mathrm{N}(C)\{0<1\} \underset{\mathrm{N}(C)\{1\}}{\times} \mathrm{N}(C)\{1<2\} \underset{\mathrm{N}(C)\{2\}}{\times} \cdots \underset{\mathrm{N}(C)\{n-1\}}{\times} \mathrm{N}(C)\{n-1<n\}$$

induced by the restriction maps $\mathrm{N}(C)[n] \to \mathrm{N}(C)\{i<i+1\}$ is a bijection of sets. (The left hand side is $\mathrm{N}(C)_n$ and the right hand side is the iterated pullback $\mathrm{N}(C)_1 \times_{\mathrm{N}(C)_0} \cdots \times_{\mathrm{N}(C)_0} \mathrm{N}(C)_1$.)

How does this relate to Question 23?

Question 33. If X is a simplicial set such that for each integer $n > 0$, the map

$$X[n] \to X\{0<1\} \underset{X\{1\}}{\times} X\{1<2\} \underset{X\{2\}}{\times} \cdots \underset{X\{n-1\}}{\times} X\{n-1<n\}$$

induced by the restriction maps $X[n] \to X\{i<i+1\}$ is a bijection of sets, is X the nerve of a category?

Question 34. Can you prove an appropriate variant of Question 32 when $\mathrm{N}(C)$ is replaced by an arbitrary ∞-category?

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