INTRODUCTION TO THE COBORDISM HYPOTHESIS AFTER HOPKINS-LURIE

ARAMINTA AMABEL

Contents

All manifolds are (smooth) oriented and compact.

Goal: Formulate the hypothesis and indicate why you might believe it.

Part 1. Modelling field theories

The notion of a field theory is captured my multiple different models, which all take various different common "features" into account.

1. Features

1.1. Space, time, and spacetime. Two examples of features are:

(1) space: where we are (e.g. a manifold X),
(2) time: how long the experiment runs (e.g. an interval I).

These two features give rise to *spacetime* $X \times I$.

Example 1. Consider a particle moving in a box. The *fields* are the paths a particle can take. I.e. for $U \subset X \times I$

$$\operatorname{Fields}(U) = \operatorname{Maps}(U, \operatorname{Box}) \ . \tag{1}$$

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Notes by: Jackson Van Dyke.

1.2. **Rules.** *Rules* are differential equations which govern which paths are allowed.

Example 2. In a free massless theory, paths are straight lines.

Rules often come in the following forms:

- an equation of motion,
- the Euler-Lagrange equation,
- an action functional.

The subset of fields consisting of solutions to these rules is denoted:

$$\mathrm{Sol}\,(U) \subset \mathrm{Fields}\,(U) \ . \tag{2}$$

1.3. **Observables.** The measurements one can take are *observables*. In general, the observables are given by:

$$\mathrm{Obs}\,(U) = \mathrm{Maps}\,(\mathrm{Sol}\,(U)\,, \mathbb{R}) \ . \tag{3}$$

Example 3. One example of an observable is the length of a path. So this would be the map sending every path in Sol to its length in $\mathbb{R}$.

1.4. **Correlation functions.** In some sense, the correlation function is the output of a TFT. Examples include a statistical measurement or a partition function in statistical mechanics.

2. Atiyah TFT

Definition 1. Let $n \in \mathbb{Z}_{>0}$. Then $\mathbf{Cob}\,(n)$ is the following symmetric monoidal category. Objects are given by oriented, closed $(n-1)$-manifolds. The morphisms from an object M to an object N are given by:

$$\left\{ \begin{array}{c} B \text{ oriented } n\text{-manifold} \\ \text{with orientation preserving diffeo.} \\ \partial B \simeq \overline{M} \sqcup N \end{array} \right\} \Big/ \left\{ \begin{array}{c} \text{orientation preserving diffeo's} \\ \text{that restricts to id on } \partial \end{array} \right\}$$

where $\overline{M}$ is M with reversed orientation.

Composition is giving by gluing cobordisms, and the symmetric monoidal structure comes from the disjoint union operation.

Warning 1. There are some collar neighborhood subtleties with this gluing.

Definition 2 (Atiyah, Segal)**.** An n-dimensional TFT is a symmetric monoidal functor

$$Z \colon \mathbf{Cob}\,(n)^{\sqcup} \to \mathbf{Vect}\,(k)^{\otimes} \ . \tag{4}$$

Example 4. $Z\,(\varphi) = k$.

In table 1, the various features in the classical context are compared to those in the Atiyah formalism in table 1. In table 1, the value of Z on a closed n-manifold Y is meant in the sense that it induces a map:

$$Z\,(Y) : \underbrace{Z\,(\varphi)}_{=k} \to \underbrace{Z\,(\varphi)}_{=k} \ . \tag{5}$$

Since $\mathbf{Cob}\,(n)$ is a category, any object M has an identity morphism given by

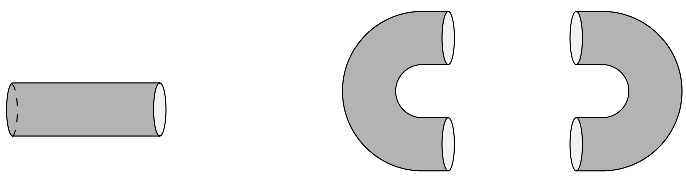
$$Z\,(M \times [0,1]) : Z\,(M) \xrightarrow{\mathrm{id}} Z\,(M) \ . \tag{6}$$

But $M \times [0,1]$ can be viewed several ways.

TABLE 1. Features of classical mechanics vs. an Atiyah TFT

Features	Classical mechanics	Atiyah TFT
Space	Fixed manifold X	*any* closed $(n-1)$-manifold
Time	$\mathbb{R}$ or I	$[0,1]$
Spacetime	$X \times I$	*any* n-manifold with boundary.
Fields	for $U \subset X \times I$ $\text{Fields} = \text{Maps}\,(U, \text{Box})$	?
Equations of motion	differential equations, $\text{Sol}\,(U) \subset \text{Fields}\,(U)$.	?
Observables	$\text{Maps}\,(\text{Sol}\,(U), \mathbb{R})$	$Z\left(S^{n-1}\right)$
Partition function	?	$Z\,(Y)$ (for n-dim closed Y)

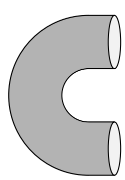
Example 5. Let $M = S^1$. Then we can think of $M \times I$ in the following ways:

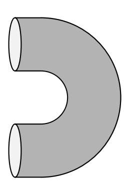
(7) 

(8) $M \to M$ $\qquad$ $\emptyset \to M \sqcup \overline{M}$ $\qquad$ $M \sqcup \overline{M} \to \emptyset$

so Z sends them to:

(9) Z () $= \mathrm{id}_{Z(M)}$

(10) Z () $: Z\,(M) \otimes Z\left(\overline{M}\right) \to k$

(11) Z () $: k \to Z\left(\overline{M}\right) \otimes Z\,(M)$.

Proposition 1. $Z\,(M)$ *is finite-dimensional, and the pairing*

$$Z\,(M) \otimes Z\left(\overline{M}\right) \to k \tag{12}$$

is a perfect pairing, so

$$Z\left(\overline{M}\right) \cong Z(M)^{\vee} \qquad\qquad Z(M) \cong Z\left(\overline{M}\right)^{\vee} \ . \tag{13}$$

Proof. We will save the proof for the discussion section. $\square$

Part 2. Classifying TFT's

3. Dimension 1

Consider a 1-dimensional field theory:

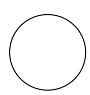
$$Z\colon \mathbf{Cob}(1)^{\sqcup} \to \mathbf{Vect}(k)^{\otimes} \ . \tag{14}$$

The objects of $\mathbf{Cob}(1)$ are oriented points. Call the positively oriented point P, and the negatively oriented point Q.

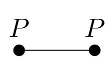
This sends P to some $V \in \mathbf{Vect}(k)$. Then by Proposition 1 we have that

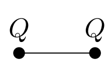
$$\begin{aligned} \mathbf{Cob}(1) &\xrightarrow{\ Z\ } \mathbf{Vect}(k) \\ P &\longmapsto V \\ Q &\longmapsto V^{\vee} \\ P \sqcup Q &\longmapsto V \otimes V^{\vee} = \mathrm{End}(V) \\ \sqcup_{S_+} P \sqcup_{S_-} Q &\mapsto \bigotimes_{S_+} V \otimes \bigotimes_{S_-} V^{\vee} \ . \end{aligned} \tag{15}$$

So V determines Z on objects. What about morphisms? These are either an interval or a circle:

(16)  .

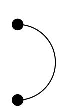
Both of the intervals get sent to the identity:

(17) $Z\Big($  $\Big) = \mathrm{id}_V$

(18) $Z\Big($  $\Big) = \mathrm{id}_{V^{\vee}}$

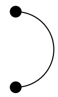
(19)

and the following:

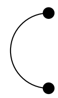
(20) 

$$P \sqcup Q \to \emptyset \tag{21}$$

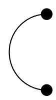
gets sent to the evaluation map:

(22) $Z\Big($  $\Big)$:
$$V \otimes V^{\vee} \longrightarrow k$$
$$(x \otimes f) \longmapsto f(x) \ .$$

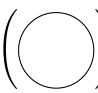
The other direction:

(23)  $: \emptyset \to P \sqcup Q$

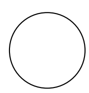
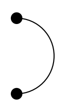
gets sent to the following map:

(24) $Z\Big($  $\Big)$:
$$k \longrightarrow \mathrm{End}(V)$$
$$\lambda \longmapsto \lambda\, \mathrm{id}_V \ .$$

The assignment to a circle is some map:

(25) $Z\Big($  $\Big) : k \to k \ .$

To understand it, we split the circle up as:

(26)  $=$  $\cup$  .

This tells us that $Z\left(S^1\right)$ factors as:

$$\begin{tikzcd} k \arrow[rr, "Z(S^1)"] \arrow[dr] & & k \\ & \mathrm{End}(V) \arrow[ur] & \end{tikzcd} \tag{27}$$

so we have that

$$Z\left(S^1\right) = \mathrm{Tr}\left(\mathrm{id}_V\right) = \dim V \ . \tag{28}$$

Upshot: V determines Z completely.

So we might hope for the functor:

$$\begin{aligned} \mathrm{Fun}^{\otimes}\left(\mathbf{Cob}(1), \mathbf{Vect}(k)\right) &\longrightarrow \mathbf{Vect}(k) \\ Z &\longmapsto Z(P) \end{aligned} \tag{29}$$

to be an equivalence.

Question 1. Is this functor surjective?

Answer. No. The image only includes finite-dimensional vector spaces.

Remark 1. The observables are $Z\left(S^0\right) = Z\left(P \sqcup Q\right) = \mathrm{End}\left(V\right)$. Note that $\mathrm{End}\left(V\right)$ is an associative algebra.

4. Dimension 2

Consider a 2-dimensional TFT:

$$Z\colon \mathbf{Cob}\left(2\right) \to \mathbf{Vect}\left(k\right) \ . \tag{30}$$

Objects of $\mathbf{Cob}\left(2\right)$ are disjoint unions of copies of S^1. Write $Z\left(S^1\right) = A$.

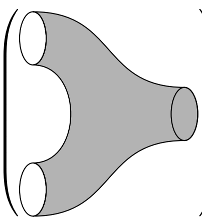
Consider a surface regarded as a morphism in $\mathbf{Cob}\left(2\right)$. We can decompose this into pairs of pants and disks. This decomposition corresponds to factoring the morphism into the morphisms attached to the pieces. So we can just calculate the value of Z on the pieces, and then composition gives us the final answer.

Explicitly: Any time we have a disjoint union of something crossed with an interval, this is just the identity on the tensor product, e.g.

$$Z\left(\left(S^1 \sqcup S^1\right) \times [0,1]\right) = \mathrm{id}_{Z\left(S^1 \sqcup S^1\right)} = \mathrm{id}_{A^{\otimes 2}} \ . \tag{31}$$

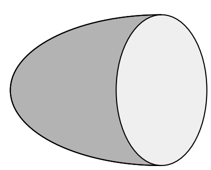
Note that $S^1 = \overline{S^1}$, since S^1 has an orientation reversing diffeomorphism.

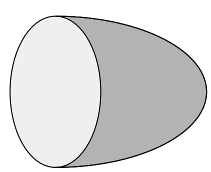
The first nontrivial piece in the decomposition of our surface is the pair of pants. Z sends this to a multiplication map on A:

(32) Z () $: A \otimes A \xrightarrow{m} A$.

Exercise 1. Check that m is a commutative and associative multiplication.

We also have the 2-disk regarded as either a morphism $\emptyset \to S^1$ or $S^1 \to \emptyset$:

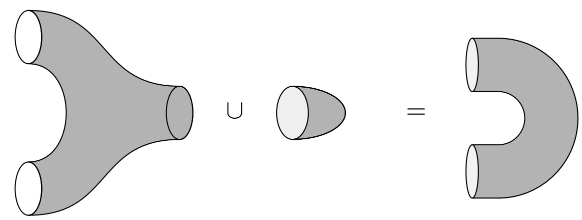
(33) Z () $: k \to A$

(34) Z () $: A \xrightarrow{\mathrm{tr}} k$.

The former case defines the unit to the multiplication m coming from the pair of pants, and the latter gets sent to a map $A \xrightarrow{\mathrm{tr}} k$ which we call tr.

Remark 2. The pants in the opposite direction (i.e. $S^1 \to S^1 \sqcup S^1$) gives a comultiplication on A, for which tr is the counit.

If we glue the disk to the "waist" of the pair of pants:

(35) 

we get a bordism which Z sends to a map

$$A \otimes A \to k \ . \tag{36}$$

Recall from Proposition 1 that this is in fact a perfect pairing.

Definition 3. A commutative *Frobenius algebra* is a finite-dimensional, commutative k-algebra with a linear map $\mathrm{tr}\colon A \to k$ such that $\mathrm{tr}\,(ab)$ is a perfect pairing.

In light of this definition, we have seen that $Z\left(S^1\right) = A$ is a Frobenius algebra. Then we might wonder if this is an equivalence, and the following folk theorem tells us that it is true.

Theorem 2 (See [Koc04])**.** *There is an isomorphism between* 2*-dimensional TFT's and commutative Frobenius algebras over* k*:*

$$\begin{aligned} \mathrm{Fuk}^{\otimes}\left(\mathbf{Cob}\,(Z)\,, \mathbf{Vect}\,(k)\right) &\xrightarrow{\sim} \mathrm{Frob}_k^{com} \\ Z &\longmapsto Z\left(S^1\right) \ . \end{aligned} \tag{37}$$

Remark 3. In dimension 2, the observables are $Z\left(S^1\right) = A$. So in this case the observables have some algebraic structure. In particular, they have this multiplication coming from the pair of pants.

Question 2. What about $Z\,(i\text{-legged pants})$? In particular, what does this give us in the higher-categorical/extended setting?

5. Higher dimensions

Consider the 3-dimensional case. $\mathbf{Cob}\,(3)$ has a lot of objects. For example, the closed surface of any genus is an object. But usually we're not scared of Σ_g because

$$\Sigma_g = \bigoplus_g T^2 \ . \tag{38}$$

So we want Z to understand this sort of decomposition.

In other words, we want to do a handlebody decomposition as we did in dimension 2. But in that case we were considering these surfaces as morphisms. So we want Z to know about gluing along S^1.

To accomplish this, we need a new category, say $\mathbf{Cob}_n\,(n)$, that knows about manifolds of all dimensions less than n (rather than just $(n-1)$-dimensional manifolds).

To do this, we need higher category theory.

Warning 2. It is hard to define higher categories, and to define this specific one.

Definition 4. An extended TFT of dimension n, valued in an n-category $\mathcal{C}$, is a symmetric monoidal functor

$$Z\colon \mathbf{Cob}_n(n) \to \mathbf{C}\ . \tag{39}$$

Our first guess might be that we have an equivalence of n-categories:

$$\begin{array}{ccc} \mathrm{Fun}^{\otimes}\left(\mathbf{Cob}_n(n), \mathbf{C}\right) & \xrightarrow{\sim} & \mathbf{C}^{f.d.} \\ Z & \longmapsto & Z(p) \end{array} . \tag{40}$$

But this is in fact false. We need to actually consider framed bordisms.

Theorem 3 (Baez-Dolan cobordism hypothesis, Hopkins-Lurie, Lurie)**.**

$$\begin{array}{ccc} \mathrm{Fun}^{\otimes}\left(\mathbf{Cob}_n^{fr}(n), \mathbf{C}\right) & \xrightarrow{\sim} & \mathbf{C}^{f.d.} \\ Z & \longmapsto & Z(p) \end{array} . \tag{41}$$

References

[Koc04] Joachim Kock. *Frobenius algebras and 2D topological quantum field theories*, volume 59 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 2004. 7