# DISCUSSION SECTION 9

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ABSTRACT. Below you'll find a collection of questions. You should think of these questions less as a homework assignment and more as a playground to run around in and spark your imagination. It's more fun playing on the swings with friends, so come to discussion section to talk with others about anything you find confusing or exciting!

CONTENTS



## 1. REVIEW

We have reached the end of the proof section of [3]. The last few weeks will be about applications and alternative proofs. The following questions are meant to remind you of some of the steps in the proof and recreate a big picture story of the Cobordism Hypothesis.

**Question 1.** What is an oriented $n$-dimension fully extended TQFT?

**Question 2.** Give a precise formulation of the Cobordism Hypothesis for oriented manifolds.

**Question 3.** What does the Cobordism Hypothesis say in the framed case?

**Question 4.** Prove the Cobordism Hypothesis for oriented manifolds in dimension 1.

**Question 5.** Lurie's outlined proof of the Cobordism Hypothesis has 5 big parts (corresponding to 5 sections of [3] and 5 talks in our seminar). What are they?

Given Eliashberg and Mishachev's theorem [1] that the space of framed functions is contractible, how can you shorten(/change) Lurie's proof?

**Question 6.** What is a Segal space? What is an $(\infty, n)$-category?

## 2. TRUNCATIONS OF SPACES

The problems in this section are meant to get you thinking about truncations of spaces, how you might construct them, and what properties the truncation functors have. Recall that for each integer $m \geq 0$, we write $\mathbf{Spc}_{\leq m} \subset \mathbf{Spc}$ for the full subcategory spanned by the *m-truncated* spaces. The inclusion $\mathbf{Spc}_{\leq m} \subset \mathbf{Spc}$ admits a left adjoint $\tau_{\leq m} \colon \mathbf{Spc} \to \mathbf{Spc}_{\leq m}$; Question 7 is about constructing this adjoint.

Given a CW complex $X$, we can always inductively add cells to $X$ to kill off all higher homotopy and replace $X$ by an $m$-truncated space with the same homotopy groups as $X$ in degrees $\leq m$. Unfortunately, this procedure involves many choices and is not very functorial.

**Question 7** (functorial $m$-truncations via Kan complexes)**.** Write down a functor $t_{\leq m} : \mathbf{sSet} \to \mathbf{sSet}$ and a natural transformation $\eta : \mathrm{id}_{\mathbf{sSet}} \to t_{\leq m}$ with the following properties:

(a) The functor $t_{\leq m}$ preserves Kan complexes.
(b) The functor $t_{\leq m}$ commutes with products.
(c) If $K$ is a Kan complex, then $t_{\leq m}(K)$ is an $m$-truncated Kan complex and the natural map $\eta_K : K \to t_{\leq m}(K)$ induces an isomorphism on homotopy groups in degrees $\leq m$.
(d) If $K$ is a Kan complex, then the natural map $K \to \lim_{m \geq 0} t_{\leq m}(K)$ is a homotopy equivalence.

After inverting homotopy equivalence of Kan complexes, can you prove that the functor you've constructed has the desired universal property of the $m$-truncation functor $\tau_{\leq m} : \mathbf{Spc} \to \mathbf{Spc}_{\leq m}$?

*Hint:* An truncation is kind of the 'opposite' of a skeleton.

**Question 8** (internal formulation of $m$-truncatedness)**.** Let us extend the definition of $m$-truncated spaces to integers $m \geq -2$ as follows: a $(-2)$-*truncated space* is a space that is contractible, and a $(-1)$-*truncated space* is a space that is empty or contractible.

For $m \geq -2$, we say that a morphism $f : X \to Y$ is *$m$-truncated* if the fibers of $f$ are $m$-truncated.

(a) What is another name for a $(-2)$-truncated morphism?
(b) Reformulate the definition of an $m$-truncated morphism in terms of homotopy groups.
(c) For $m \geq -1$, show that a morphism $f : X \to Y$ is $m$-truncated if and only if the diagonal morphism
$$\Delta_f : X \to X \times_Y X$$
is $(n-1)$-truncated.

**Question 9** (properties of $m$-truncation)**.** Let $f : X \to Z$ and $g : Y \to Z$ be morphisms in $\mathbf{Spc}$.

(a) For each $m \geq 0$, provide an example that shows that the natural map
$$\tau_{\leq m}(X \times_Z Y) \to \tau_{\leq m}(X) \times_{\tau_{\leq m}(Z)} \tau_{\leq m}(Y)$$
need not be an equivalence.
(b) For each $m \geq 0$, show that if $Z$ is $m$-truncated, then the natural map
$$\tau_{\leq m}(X \times_Z Y) \to \tau_{\leq m}(X) \times_Z \tau_{\leq m}(Y)$$
is an equivalence.

*Hint:* Use Question 8.

## 3. Truncations of $(\infty, n)$-categories

**Question 10** (truncations of $(\infty, 1)$-categories)**.** In the complete Segal space model for $(\infty, 1)$-categories, a complete Segal space $C : \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Spc}$ is an *$(m, 1)$-category* if all of the values of $C$ are $m$-truncated spaces.

(a) Using the complete Segal space model for $(\infty, 1)$-categories, prove that the inclusion of $(m, 1)$-categories into $(\infty, 1)$-categories admits a left adjoint $\tau_{\leq m}$.
(b) Is this left adjoint just given by pointwise application of the $m$-truncation functor $\mathbf{Spc} \to \mathbf{Spc}_{\leq m}$? If not, are there some instances where it is?
(c) Prove that for any $(\infty, 1)$-category $C$, the natural morphism $C \to \lim_{m \geq 0} \tau_{\leq m}(C)$ is an equivalence in $\mathbf{Cat}_{(\infty,1)}$.

**Question 11** (truncations of $(\infty, n)$-categories)**.** For $n \geq 2$, use the $n$-fold complete Segal space model for $(\infty, n)$-categories to:

(a) Show that the inclusion of $(m, n)$-categories into $(\infty, n)$-categories admits a left adjoint.
(b) Show that every $(\infty, n)$-category is the limit of its Postnikov tower.

## 4. Local systems on $(\infty, n)$-categories

**Question 12** (defining local systems)**.** Try to make the definition of a local system on an $(\infty, n)$-category [3, Definition 3.5.10] more precise. What's a precise way to encode all of the relations that the maps $m_{x,y,z}$ are supposed to satisfy? Start from $n = 1$ and work from there.

**Question 13** (a better way to package local systems on an $(\infty, 1)$-category)**.** Recall that a local system on an $(\infty, 0)$-category $X$ is just a functor $X \to \mathbf{Ab}$, equivalently, a functor $\tau_{\leq 1} X \to \mathbf{Ab}$. So [3, Definition 3.5.10] says that to define a local system on an $(\infty, 1)$-category $C$, we need to define functors $\mathrm{Map}_C(x, y) \to \mathbf{Ab}$ for each pair of objects $x, y \in C$, as well as provide some additional structure related to composition of morphisms in $C$. Is there a clean way of packaging this data in terms of functors out of some $(\infty, 1)$-category naturally associated to $C$ satisfying some list of properties?

Maybe a natural place to start is with the *twisted arrow* $(\infty, 1)$-*category* $\mathrm{TwAr}(C)$. The forgetful functor $\mathrm{TwAr}(C) \to C^{\mathrm{op}} \times C$ is the left fibration classifying the functor $\mathrm{Map}_C(-,-)\colon C^{\mathrm{op}} \times C \to \mathbf{Spc}$. Under straightening/unstratightening, a natural transformation from the functor $\mathrm{Map}_C(-,-)$ to the constant functor $C^{\mathrm{op}} \times C \to \mathbf{Cat}_{(\infty,1)}$ at $\mathbf{Ab}$ corresponds to a morphism of cocartesian fibrations from $\mathrm{TwAr}(C)$ to the cocartesian fibration classified by the constant functor $C^{\mathrm{op}} \times C \to \mathbf{Cat}_{(\infty,1)}$ at $\mathbf{Ab}$.

The most simple nontrivial $(n, n)$-category is the *$n$-cell* $\mathrm{Cell}_n$. The 0-cell is just the trivial category with one object with an identity morphism. For $n > 0$, the $n$-cell has two objects 0 and 1, with two non-identity 1-morphisms $a_0, a_1 \colon 0 \to 1$, two non-identity 2-morphisms $b_0, b_1 \colon a_0 \to a_1$, etc., and at the top level one $n$-morphism between the two non-identity $(n-1)$-morphisms. So $\mathrm{Cell}_1$ is the walking arrow $0 \to 1$ and $\mathrm{Cell}_2$ is the $(2, 2)$-category

$$\begin{tikzcd} 0 \arrow[r, bend left, "a_0"] \arrow[r, bend right, "a_1"'] \arrow[r, phantom, "\Downarrow"] & 1 \end{tikzcd} .$$

**Question 14** (local systems on the $n$-cell)**.** Given an explicit description of the data of a local system on:
(a) The 0-cell.
(b) The 1-cell.
(c) The 2-cell.
(d) The 3-cell.
(e) The $n$-cell for $n > 3$.

## 5. The end of the proof of the cobordism hypothesis

Recall that we write $\mathbf{Bord}_n^{\mathrm{ff}}$ for the $(\infty, n)$-category of cobordisms of $n$-manifolds along with specified framed function data constructed in Dylan's talk (see [3, p. 37]). There is a natural functor

$$f\colon \mathbf{Bord}_n^{\mathrm{ff}} \to \mathbf{Bord}_n$$

that forgets the framed function data.

**Question 15.** Let $\mathcal{A}$ be a local system on the $(\infty, n)$-category $\mathbf{Bord}_n$.
(a) Go through the details on [3, pp. 85–86] of the construction of the local system $\mathcal{B}$ on $\mathrm{BO}(n)$ associated to the local system $f^*\mathcal{A}$ on $\mathbf{Bord}_n^{\mathrm{ff}}$. (This local system is denoted by $\mathrm{L}_{\mathcal{A}}$ in Peter's notes.)
(b) Go through the details on [3, pp. 85–86] to show that there is a canonical isomorphism

$$\mathrm{H}^m_{\otimes}(\mathbf{Bord}_n^{\mathrm{ff}}, \mathbf{Bord}_{n-1}; f^*\mathcal{A}) \simeq \mathrm{H}^{m-n}(\mathrm{BO}(n); \mathcal{B}) .$$

**Question 16** (the cobordism hypothesis and Galatius–Madsen–Tillmann–Weiss Theorem)**.** The purpose of this problem is to get you thinking about [3, Remark 3.5.24].

(a) Review the definition of the Madsen–Tillmann spectra $\mathrm{MTO}(k)$.

(b) Review the Galatius–Madsen–Tillmann–Weiss Theorem on the homotopy type of cobordism categories: $|\mathbf{Bord}_k| \simeq \Omega^\infty \Sigma^k \mathrm{MTO}(k)$.

(c) Try to prove that there is a fiber sequence of spectra

$$\Sigma^{n-1}\,\mathrm{MTO}(n-1) \to \Sigma^n\,\mathrm{MTO}(n) \to \Sigma_+^{\infty+n}\,\mathrm{BO}(n)\,.$$

If you get stuck, look at the proof of [2, Proposition 3.1].

(d) Deduce the unoriented version of the cobordism hypothesis from the above fiber sequence.

(e) Can you come up with an appropriate generalization of the results of Galatius–Madsen–Tillmann–Weiss that would imply the general version of the cobordism hypothesis?

## References


[1] Y. M. Eliashberg and N. M. Mishachev. The space of framed functions is contractible. In *Essays in mathematics and its applications*, pages 81–109. Springer, Heidelberg, 2012.

[2] Søren Galatius, Ulrike Tillmann, Ib Madsen, and Michael Weiss. The homotopy type of the cobordism category. *Acta Math.*, 202(2):195–239, 2009.

[3] Jacob Lurie. On the classification of topological field theories. In *Current developments in mathematics, 2008*, pages 129–280. Int. Press, Somerville, MA, 2009.