DISCUSSION SECTION 7

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Abstract. Below you'll find a collection of questions. You should think of these questions less as a homework assignment and more as a playground to run around in and spark your imagination. It's more fun playing on the swings with friends, so come to discussion section to talk with others about anything you find confusing or exciting!

Contents

1. Categorical Chain Complexes and Skeletal Sequences

Question 1. Let $(\Omega_d)_{de\geq 0}$ be the oriented bordism groups,

$$\Omega_d = \{\text{closed oriented } d\text{-manifolds}\}/\{\text{oriented bordism}\}$$

Define a chain complex (of abelian groups, not categorical) with homology $(\Omega_d)_{d\geq 0}$.

See the bottom of page 61 of [1] for an answer.

For the definition of a categorical chain complex, see [1, Defn. 3.3.6].

Question 2. Can you recover the definition of a chain complex of abelian groups from the definition of a categorical chain complex?

For a definition of $\mathrm{Cob}^{\mathrm{un}}_{\partial}(k)$ see [1, Defn. 3.3.1].

Question 3. Show that $\{\mathrm{Cob}^{\mathrm{un}}_{\partial}(k), \partial\}$ is a categorical chain complex.

Question 4. Read [1, Defn. 3.3.8] of a functor $f\colon \mathcal{C} \to \mathcal{D}$ being k-connective.

- For spaces $X \subseteq Y$, when is the induced functor
$$f\colon \pi_{\leq\infty} X \to \pi_{\leq\infty} Y$$
k-connective?
- Show that $\mathbf{Bord}_k \to \mathbf{Bord}_{k+1}$ is k-connective.

See Examples 3.3.9 and 3.3.10 of [1] for answers to the above. The following is Example 3.3.13 and Warning 3.3.15 of [1].

Question 5. Let $\mathcal{B}$ be a symmetric monoidal (∞, n)-category with duals. Construct a canonical skeletal sequence from $\mathcal{B}$ by throwing away non-invertible morphisms.

- . Is the skeletal sequence

$$\mathbf{Bord}_1 \to \cdots \to \mathbf{Bord_n}$$

an example of such a canonical skeletal sequence for some $\mathcal{B}$?

Question 6. Can you describe a categorical chain complex as some sort of associated graded object?

2. Grothendieck Construction

Question 7. Recall the definition of a lax symmetric monoidal functor. Let $F\colon \mathcal{B}_1 \to \mathcal{B}_2$ be a symmetric monoidal functor. Show that the functor $M_F\colon \mathcal{B}_1 \to \mathsf{Cat}_{(\infty,1)}$ defined in [1, Not. 3.3.19] given by

$$M_F(X) = \mathrm{Maps}_{\mathcal{B}_2}(\mathbb{1}, F(X))$$

is lax symmetric monoidal.

Recall Construction 3.3.23 of [1] of the Grothendieck construction.

Question 8. Given $\mathcal{B}$ an $(\infty,1)$-category and $M\colon \mathcal{B} \to \mathsf{Cat}_{(\infty,1)}$ a functor, show that the projection $\pi\colon \mathrm{Groth}(\mathcal{B}, M) \to \mathcal{B}$ is a coCartesian fibration.

Question 9. What is the relationship between the Grothendieck construction for ∞-categories and the straightening/unstraightening principal?

Question 10. This quest is to understand simple examples of the Grothendieck construction, $\mathrm{Groth}(\mathcal{B}, M)$ for various choices of $\mathcal{B}$ and $M\colon \mathcal{B} \to \mathsf{Cat}_{(\infty,1)}$.

- Take $\mathcal{B}$ to be pt, the ∞-category with one object and only the identity morphism. For an arbitrary functor

  $$M\colon \mathrm{pt} \to \mathsf{Cat}_{(\infty,1)},$$

  describe $\mathrm{Groth}(\mathrm{pt}, M)$.
- Take $\mathcal{B}$ to be an arbitrary ∞-category and M to be the constant functor at some fixed ∞-category $\mathcal{D}$. Describe $\mathrm{Groth}(\mathcal{B}, M)$.

  See [2, Ex. 1.15] for an answer.
- Take $\mathcal{B} = [1]$ the category with two objects and one non-identity morphism. Let $M\colon [1] \to \mathsf{Cat}_{(\infty,1)}$ sending the non-identity morphism to the functor

  $$f\colon \mathcal{C}_0 \to \mathcal{C}_1$$

  Describe $\mathrm{Groth}([1], F)$.

  See [2, Ex. 1.8] for an answer.
- Let G and H be groups. Let $\mathcal{C}_G$ and $\mathcal{C}_H$ denote the corresponding categories with a single object $*_G$ and $*_H$, respectively. A functor

  $$M\colon \mathcal{C}_G \to \mathsf{Cat}_{(\infty,1)}$$

  sending $*_G$ to $\mathcal{C}_H$ is determined by what extra data? Given such a functor, what is $\mathrm{Groth}(\mathcal{C}_G, M)$?

  I took this from the Wikipedia page on the Grothendieck construction; you can find an answer there.

Question 11. Can you describe the Grothendieck construction in terms of more familiar categorical constructions such as coends and colimits?

3. Index Filtration Prep

Question 12. What is a Morse function? Look up or recall the definition of critical points and their index.

The Wikepedia page for Handle decomposition is one place to look. You could also look at Milnor's Morse theory book, or his Lectures on the h-cobordism theorem.

Question 13. Write the definition of a handle attachment of index k onto a manifold of dimension n. What does a handle attachment of index n look like? What about of index 0?

Question 14. Draw the height function h for the circle S^1.

- Show that h is a Morse function.
- What are the critical points of h?
- What index do the critical points of h have?
- Show how h decomposes S^1 into a sequence of handle attachments.
- What happens when we do the same for S^2 instead of S^1? What about S^n?

Question 15. Draw the height function h for a torus T.

- Show that h is a Morse function.
- What are the critical points of h?
- What index do the critical points of h have?
- Show how h decomposes T into a sequence of handle attachments.
- For either S^1 or S^2, construct (via drawing) a different Morse function that has a different number of critical points than h does. Draw the corresponding handle decomposition.

In particular, note that different Morse functions can lead to different handle decompositions.

Question 16. Fix a manifold M. Consider the set of all Morse functions on M. Think of a reasonable way to topologize this set of functions. (Remember that there are various ways to topologize the space of all smooth functions). Is this *space* of Morse functions connected?

Hint: Can you make a path between Morse functions with different numbers of critical points?

Question 17. Let Σ be a surface of genus 2.

- Show that the height function h is a Morse function.
- Draw the handle body decomposition of Σ corresponding to h.
- Draw a series of pictures depicting a different handle presentation of Σ where the handles of index k are attached simultaneously.

I stole this from a homework problem John Francis gave.

References

[1] Jacob Lurie. On the classification of topological field theories. In *Current developments in mathematics, 2008*, pages 129–280. Int. Press, Somerville, MA, 2009.

[2] Aaron Mazel-Gee. All about the Grothendieck construction. *arXiv e-prints*, page arXiv:1510.03525, October 2015.