DISCUSSION SECTION 6

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Abstract. Below you'll find a collection of questions. You should think of these questions less as a homework assignment and more as a playground to run around in and spark your imagination. It's more fun playing on the swings with friends, so come to discussion section to talk with others about anything you find confusing or exciting!

Question 1. Consider $\mathcal{F}\colon X \times Y \to \mathrm{Cat}_{(\infty,n-1)}$ defined in Lucy's talk and in (3) on page 58 of [1]. It sends a pair (x, y) to

$$\mathcal{F}_{x,y} := \mathrm{Fam}_{n-1}\mathsf{Maps}_{\mathcal{C}}((f(x), g(y))$$

What are global sections of $\mathcal{F}$ and how do they form an $(\infty, n-1)$-category?

Question 2. What does Fam_n look like for small values of n?

- Show/convince yourself that the morphisms are indeed given by correspondences.
- Informally, what are the k morphisms of Fam_n for $k \leq n$ and $k > n$?
- What is its symmetric monoidal structure?

Question 3. Take $\mathcal{C}$ to be the $(\infty, 1)$-category of chain complexes of finite-dimensional vector spaces with direct sum. Describe the objects and 1-morphisms of $\mathrm{Fam}_1(\mathcal{C})$. What is the symmetric monoidal structure on $\mathrm{Fam}_1(\mathcal{C})$?

Question 4. How would one rigorously show that the (∞, n)-category $\mathrm{Fam}_n(\mathcal{C})$ is symmetric monoidal if $\mathcal{C}$ is?

Question 5. Show that $\mathrm{Fam}_n(\mathcal{C})$ has duals if $\mathcal{C}$ does.

Question 6. Show explicitly that Fam_2 has duals.

Question 7. The cobordism hypothesis predicts that $O(n)$ acts on Fam_n.

- What does this action look like for $n = 1$?
- How about $n = 2, 3, ..$?

Question 8. What are objects of $(\mathrm{Fam}_n^{\sim})^{hO(n)}$? Associate to any such object a pair (X, ζ), where X is a topological space and ζ is an n-dimensional vector bundle on X with an inner product

See the discussion at the bottom of page 58 of [1] for an answer.

Question 9. Look up and read about the straightening-unstraightening principle/equivalence that Lucy mentioned last week. Give a precise statement of it.

Question 10. Use the straightening/unstraightening equivalence to give a rigorous proof of Proposition 3.2.7 of [1].

Question 11. Describe/give a model for $B_{(B,\xi)}(M) = Z_{(B,\xi)}(M)$.

Recall the pointed version Fam_n^* of Fam_n (see page 59 of [1] for a reminder).

Question 12. We can think of $\mathrm{Fam}_n^*(\mathcal{C}) \to \mathrm{Fam}_n(\mathcal{C})$ as a 'universal bundle' for what?

Question 13. In step (a) of the proof Lucy gave last week, why did we require $\overline{Z}$ to lift $Z_{(B,\xi)}$?

Question 14. Why is the restriction of $Z_{(B,\xi)}(M)$ to $\mathbf{Bord}_{n-1}$ equivalent to the composite

$$\mathbf{Bord}_{n-1} \xrightarrow{Z_{(B_0,\xi_0)}} \mathrm{Fam}_{n-1} \to \mathrm{Fam}_n?$$

Which property of (B_0, ξ_0) is used here?

References

[1] Jacob Lurie. On the classification of topological field theories. In *Current developments in mathematics, 2008*, pages 129–280. Int. Press, Somerville, MA, 2009.